

# 2-Resonant fullerenes\*

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## Abstract

A *fullerene graph*  $F$  is a planar cubic graph with exactly 12 pentagonal faces and other hexagonal faces. A set  $\mathcal{H}$  of disjoint hexagons of  $F$  is called a *resonant pattern* (or *sextet pattern*) if  $F$  has a perfect matching  $M$  such that every hexagon in  $\mathcal{H}$  is  $M$ -alternating.  $F$  is said to be  $k$ -*resonant* if any  $i$  ( $0 \leq i \leq k$ ) disjoint hexagons of  $F$  form a resonant pattern. It was known that each fullerene graph is 1-resonant and all 3-resonant fullerenes are only the nine graphs. In this paper, we show that the fullerene graphs which do not contain the subgraph  $L$  or  $R$  as illustrated in Fig. 1 are 2-resonant except for the specific eleven graphs. This result implies that each IPR fullerene is 2-resonant.

**Key words:** Fullerene graph; Perfect matching; Resonant pattern

## 1 Introduction

A *fullerene graph* is a 3-connected planar cubic graph which has exactly 12 pentagonal faces and other hexagonal faces. Such graphs are suitable models for fullerene molecules: carbon atoms are represented by vertices, whereas edges represent chemical bonds between two atoms. It is well known that a fullerene graph on  $n$  vertices exists for any even  $n \geq 20, n \neq 22$  [9]. Since the discovery of the first fullerene molecule  $C_{60}$  [14] in 1985, the fullerenes have pioneered a new field of study. Various properties of fullerene graphs were investigated from both chemical and mathematical points of view [1, 5, 6, 7, 9, 11, 12, 25, 26].

Let  $F$  be a fullerene graph. A *perfect matching* (or Kekulé structure) of  $F$  is a set of disjoint edges  $M$  such that every vertex of  $F$  is incident with an edge in  $M$ . It has

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been shown [12] that fullerene graphs have exponentially many perfect matchings. A set  $\mathcal{H}$  of mutually disjoint hexagons is called a *resonant pattern* (or *sextet pattern*) if  $F$  has a perfect matching  $M$  such that every hexagon in  $\mathcal{H}$  is  $M$ -alternating. A fullerene graph  $F$  is  $k$ -*resonant* (or  $k$ -*coverable*,  $k \geq 1$ ) if any  $i$  ( $0 \leq i \leq k$ ) disjoint hexagons of  $F$  form a resonant pattern. The concept of resonance originates from Clar's aromatic sextet theory [3] and Randić's conjugated circuit model [18, 19]. The  $k$ -resonance of many types of graphs, including benzenoid graphs, toroidal and Klein-bottle fullerenes, boron-nitrogen fullerene graphs and (3,6)-fullerene graphs, were investigated extensively [2, 16, 21, 22, 24, 27, 28].

In [22] Ye et al. showed that every fullerene graph is 1-resonant and there are exactly nine fullerene graphs  $F_{20}, F_{24}, F_{28}, F_{32}, F_{36}^1, F_{36}^2, F_{40}, F_{48}, F_{60}$  which are also  $k$ -resonant for each  $k \geq 3$ , but not all fullerene graphs are 2-resonant. They also proved that every leapfrog fullerene graph is 2-resonant and asked a problem: whether a fullerene graph satisfying the isolated pentagon rule (IPR) is 2-resonant. In [10], Kaiser et al. gave a positive answer to the problem.



Figure 1. The subgraphs  $L, R$ .

In this paper, we consider fullerene graphs which are allowed to have some adjacent pentagons, i.e. violating IPR. Two substructures  $L$  and  $R$  consisting of four and three pentagons are defined in Fig. 1. We characterize fullerene graphs without substructures  $L$  and  $R$  which are 2-resonant and obtain the following main theorem.

**Theorem 1.1.** *Let  $F$  be a fullerene graph which does not contain the subgraph  $L$  or  $R$ . Then  $F$  is 2-resonant except for the eleven fullerene graphs in Fig. 2.*

It is easy to verify that the eleven fullerene graphs depicted in Fig. 2 are not 2-resonant since the two grey hexagons do not form a resonant pattern.

A fullerene graph is said to be IPR if it satisfies the isolated pentagon rule (IPR) (i.e. any pentagons are disjoint). It is obvious that every IPR fullerene graph has no substructures  $L$  or  $R$  and each graph in Fig. 2 has at least a pair of adjacent pentagons. So Theorem 1.1 implies immediately the following known result.

**Corollary 1.2.** *[10] Every IPR fullerene graph is 2-resonant.*

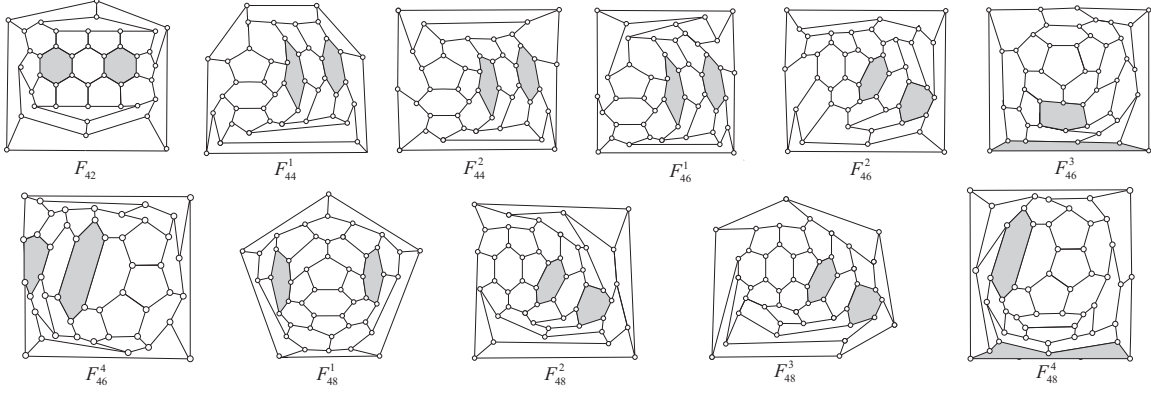


Figure 2. The eleven non-2-resonant fullerene graphs without subgraph  $L$  or  $R$ .

## 2 Definitions and preliminaries

Let  $G$  be a connected plane graph with vertex-set  $V(G)$  and edge-set  $E(G)$ . For  $X, Y \subset V(G)$ , we define  $E(X, Y)$  the set of edges having one end-vertices in  $X$  and the other in  $Y$ . We simply write  $\nabla(X)$  for  $E(X, \overline{X})$  where  $\overline{X} = V(G) - X$ . For subgraphs  $H$  and  $H'$  of  $G$ , we also simply write  $\nabla(H)$  for  $\nabla(V(H))$ , and  $E(H, H')$  for  $E(V(H), V(H'))$ ; We call  $H$  is *incident* to  $H'$  if  $V(H) \cap V(H') = \emptyset$  and  $E(H, H') \neq \emptyset$ . For a face  $f$  of  $G$ , its boundary is a closed walk and we often represent it by its boundary if unconfused. Pentagonal and hexagonal faces are referred to simply as pentagons and hexagons. Use  $\partial(G)$  to denote the boundary of the exterior face of  $G$ .

A graph  $G$  is called *factor-critical* if  $G - v$  has a perfect matching for every vertex  $v \in V(G)$ . A factor-critical graph is *trivial* if it consists of a single vertex.

**Observation 2.1.** [10] *Every non-trivial factor-critical subgraph of a fullerene graph is 2-connected.*

We call a vertex set  $S \subseteq V(G)$  *matchable* to  $\mathcal{C}_{G-S}$  if the (bipartite) graph  $G_S$ , which arises from  $G$  by contracting the components  $C \in \mathcal{C}_{G-S}$  to single vertices and deleting all the edges inside  $S$ , contains a matching of  $S$ . The following critical theorem ([4], Theorem 2.2.3) may be viewed as a strengthening of Tutte's 1-factor theorem [20].

**Theorem 2.2.** *Every graph  $G$  contains a vertex set  $S$  with the following two properties:*

- (1)  $S$  is matchable to  $\mathcal{C}_{G-S}$ ;
- (2) Every component of  $G - S$  is factor-critical.

*Given any such set  $S$ , the graph  $G$  contains a perfect matching if and only if  $|S| = |\mathcal{C}_{G-S}|$ .*

An *edge-cut* of a connected graph  $G$  is a set of edges  $C \subset E(G)$  such that  $G - C$  is disconnected. An edge-cut  $C$  of  $G$  is *cyclic* if each component of  $G - C$  contains a cycle. A

graph  $G$  is *cyclically  $k$ -edge-connected* if  $G$  cannot be separated into two components, each containing a cycle, by removing less than  $k$  edges. The *cyclical edge-connectivity* of  $G$ , denote by  $c\lambda(G)$ , is the greatest integer  $k$  such that  $G$  is cyclically  $k$ -edge-connected. For a fullerene graph  $F$ , T. Došlić [8], and Qi and Zhang [17] proved that  $c\lambda(F) = 5$ ; F. Kardoš and R. Škrekovski [11] obtained the same result by three operations on cyclic edge-cuts. There are at least twelve cyclic 5-edge-cuts—formed by the edges pointing outward each pentagonal face and there are also cyclic 6-edge-cuts—formed by the edges pointing outward each hexagonal face. These cyclic 5- and 6-edge-cuts are called *trivial*. A cyclic edge-cut  $C$  of a fullerene graph  $F$  is *non-degenerated* if both components of  $F - C$  contain precisely six pentagons. Otherwise,  $C$  is *degenerated*. Obviously, the trivial cyclic edge-cuts are degenerated.

F. Kardoš and R. Škrekovski [11], and K. Kutnar and D. Marušič [15] independently gave the nanotube structure of fullerene graphs with a non-trivial 5-cyclic edge-cut.

**Theorem 2.3.** [11, 15] *A fullerene graph has non-trivial cyclic 5-edge-cuts if and only if it is isomorphic to the graph  $G_k$  for some integer  $k \geq 1$ , where  $G_k$  is the fullerene graph comprised of two caps formed of six pentagons joined by  $k$  layers of hexagons (see Fig. 3).*

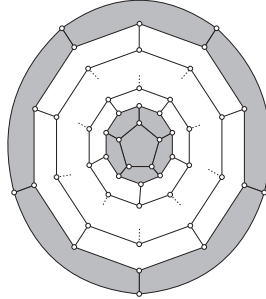


Figure 3. The graph  $G_k$  are the only fullerene graphs with non-trivial cyclic 5-edge-cuts.

Further F. Kardoš and R. Škrekovski listed the degenerated cyclic 6-edge-cuts in fullerene graphs.

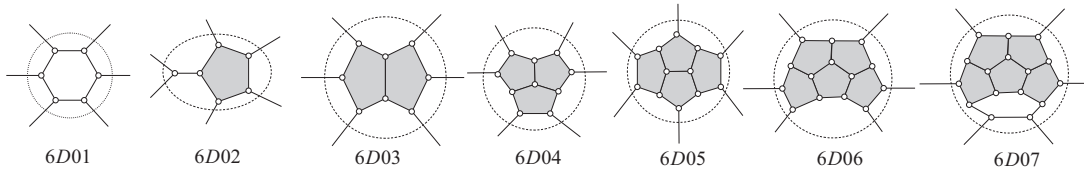


Figure 4. Degenerated cyclic 6-edge-cuts.

**Theorem 2.4.** [11] *There are precisely seven non-isomorphic graphs that can be obtained as components of degenerated cyclic 6-edge-cuts with less than six pentagons (see Fig. 4).*

Recently, F. Kardoš et al. characterized the degenerated cyclic 7-edge-cuts in fullerene graphs.

**Theorem 2.5.** [13] *There exist 57 non-isomorphic graphs that can be obtained as components of degenerated cyclic 7-edge-cuts with less than six pentagons (see Fig. 5).*

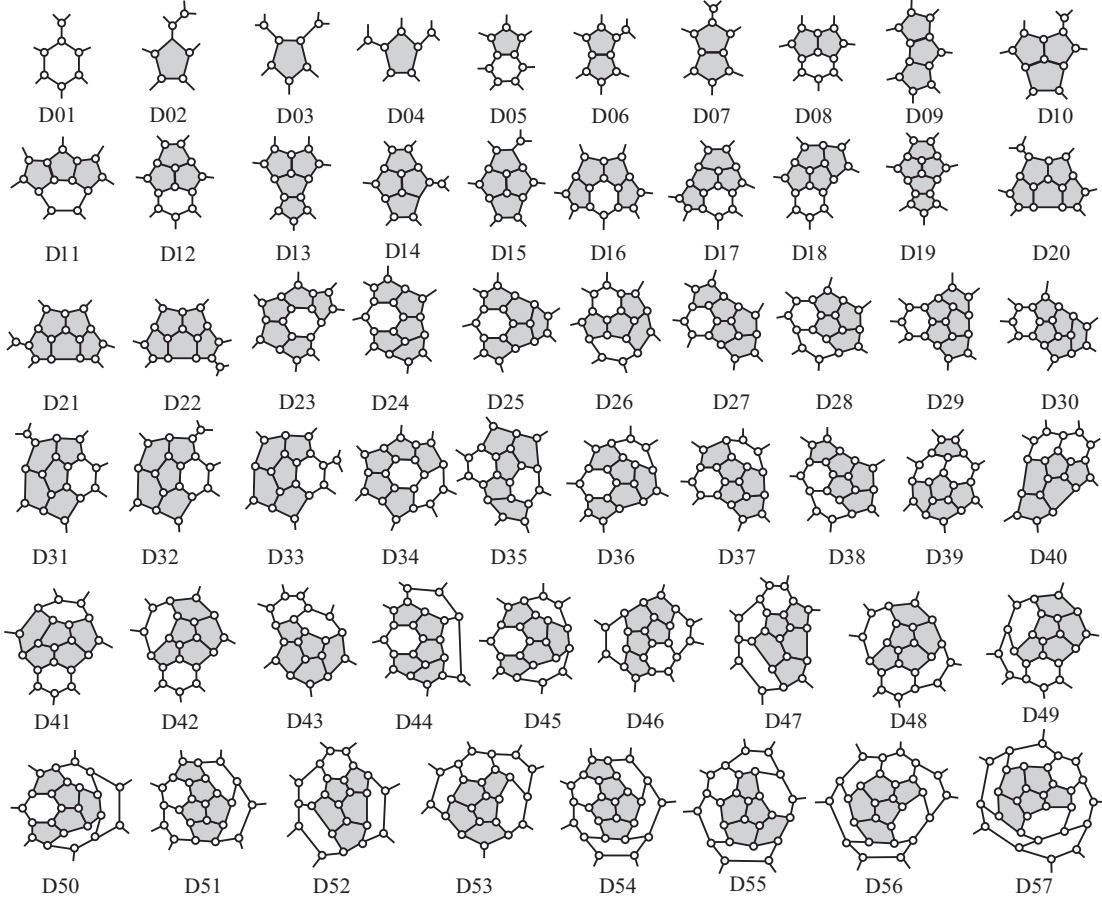


Figure 5. Degenerated cyclic 7-edge-cuts.

Their characterizations are based on the following result.

**Theorem 2.6.** [11, Theorem 1] *The cyclic edge-cuts of a fullerene graph can be constructed from the trivial ones using the reverse operations of  $(O_1)$ ,  $(O_2)$  and  $(O_3)$ .*

Here the three operations can be presented as follows (see Fig. 6 for an illustration).

$(O_1)$  If a component  $H$  contains a vertex of degree one, then using  $(O_1)$  one can modify the  $k$ -edge-cut  $C$  into a  $(k - 1)$ -edge-cut  $C_1$ .

$(O_2)$  If a component  $H$  contains two adjacent vertices of degree two, then using  $(O_2)$  one can modify the  $k$ -edge-cut  $C$  into a  $k$ -edge-cut  $C_2$ .

$(O_3)$  If the vertices of the outer faces of  $H$  are consecutively of degree 2 and 3, then using  $(O_3)$  one can modify the  $k$ -edge-cut  $C$  into a  $k$ -edge-cut  $C_3$ .

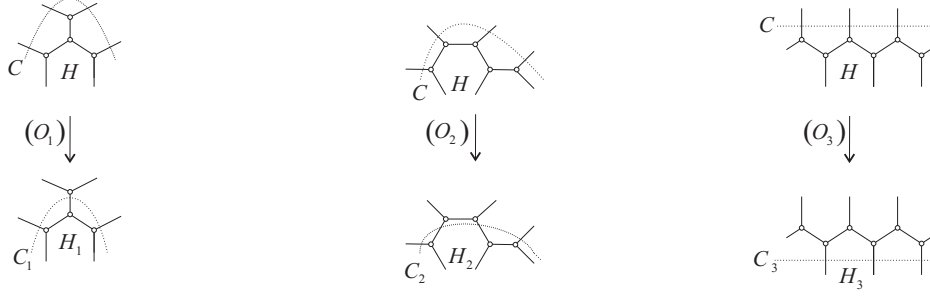


Figure 6. Three operations  $O_1$ ,  $O_2$  and  $O_3$ .

Let  $G$  be a subgraph of a fullerene graph  $F$ . A face  $f$  of  $F$  is a *neighboring face* of  $G$  if  $f$  is not a face of  $G$  and  $f$  has at least one edge in common with  $G$ . For two faces  $f_1, f_2$  of a fullerene graph  $F$ , we always say  $f_1$  *intersects*  $f_2$  if  $f_1$  is a neighboring face of  $f_2$ . The following result is known.

**Lemma 2.7.** ([27], Lemma 4.2) *Let  $H$  be a 3-regular plane graph. If  $H$  is cyclically 4-edge-connected, then there are neither three faces which are pairwise adjacent but do not share a common vertex, nor two faces which share more than one disjoint edges in  $H$ .*

**Lemma 2.8.** *Let  $f$  and  $f'$  be two disjoint faces of a fullerene graph  $F$  with  $E(f, f') = \emptyset$ . Then there is at most one common neighboring face of both  $f$  and  $f'$ .*

*Proof.* To the contrary, suppose at least two such neighboring faces exist, say  $f_1, f_2$ . Then the edge set  $C = \{e_1, e_2, e_3, e_4\}$  forms an edge cut, where  $e_1 = \partial(f) \cap \partial(f_1)$ ,  $e_2 = \partial(f) \cap \partial(f_2)$ ,  $e_3 = \partial(f') \cap \partial(f_2)$ ,  $e_4 = \partial(f') \cap \partial(f_1)$ . Since  $E(f, f') = \emptyset$ ,  $f_1, f_2$  are hexagons. On the other hand, if one component  $H$  of  $F - C$  contains a vertex  $v$  of degree one in  $H$ , then exactly two edges incident with  $v$  belong to the cut  $C$ , say  $e_1, e_2$ . Let the third edge incident with  $v$  be  $e$ . Then  $C_1 = C \setminus \{e_1, e_2\} \cup \{e\}$  is a cyclic 3-edge-cut in  $F$  since  $E(f, f') = \emptyset$  (see Fig. 7(a)), contradicting that  $c\lambda(F) = 5$ . If both components of  $F - C$  are of minimum degree two, then  $C$  is a cyclic 4-edge-cut in  $F$  (see Fig. 7(b)), again contradicting that  $c\lambda(F) = 5$ .  $\square$

**Lemma 2.9.** *Let  $f, f'$  be two faces of a fullerene graph  $F$ . Then  $|\nabla(f) \cap \nabla(f')| \leq 1$ .*

*Proof.* Suppose to the contrary that there exist two edges  $e, e'$  of  $F$  satisfying  $e, e' \in \nabla(f) \cap \nabla(f')$ . Then both  $e$  and  $e'$  are contained in a neighboring face of  $f$  and  $f'$ , say  $f_1, f_2$ , respectively. Again a cyclic edge-cut of size less than five can be gained (see Fig. 7(c)), also a contradiction.  $\square$

**Lemma 2.10.** *Assume that a fullerene graph  $F$  has no non-trivial cyclic 5-edge-cut, disjoint faces  $h_1$  and  $h_2$  of  $F$  are not incident and share a common neighboring face  $f$ . Let  $f_1, f_2$  be*

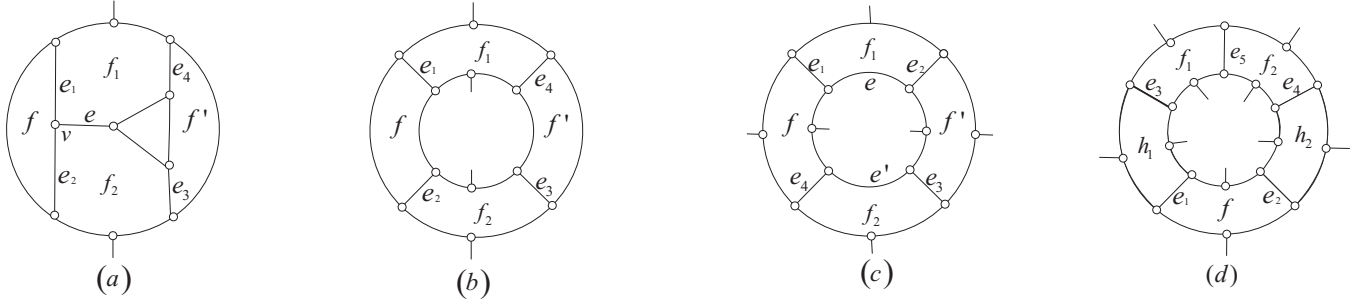


Figure 7. (a) Cyclic 3-edge-cut  $\{e, e_3, e_4\}$ , (b) and (c) two cyclic 4-edge-cuts  $\{e_1, e_2, e_3, e_4\}$ , and (d) cyclic 5-edge-cut  $\{e_1, e_2, e_3, e_4, e_5\}$ .

the other neighboring faces of  $h_1, h_2$  (respectively) different from  $f$  and  $V(f_i) \cap V(f) = \emptyset$  for some  $i \in \{1, 2\}$ . Then  $V(f_1) \cap V(f_2) = \emptyset$ .

*Proof.* Let  $E(h_1) \cap E(f) = \{e_1\}, E(h_2) \cap E(f) = \{e_2\}$ . Since  $h_1$  and  $h_2$  are not incident,  $f$  is a hexagon. By Lemma 2.7 we can know that  $f_1$  intersects  $h_1$  at exactly one edge, say  $E(h_1) \cap E(f_1) = \{e_3\}$ . Similarly,  $E(h_2) \cap E(f_2) = \{e_4\}$ . Moreover,  $f_2$  ( $f_1$ ) can not intersect  $h_1$  ( $h_2$ ) by Lemma 2.8. Suppose  $V(f_1) \cap V(f_2) \neq \emptyset$ . Then again by Lemma 2.7 we can set  $\{e_5\} = E(f_1) \cap E(f_2)$ . Since  $V(f_i) \cap V(f) = \emptyset$  for  $i \in \{1, 2\}$ ,  $C = \{e_1, \dots, e_5\}$  forms a cyclic 5-edge-cut (see Fig. 7(d)). Thus it is a trivial one, contradicting that  $h_1$  and  $h_2$  are not incident.  $\square$

A *fragment*  $B$  of a fullerene graph  $F$  is a subgraph of  $F$  consisting of a cycle together with its interior. A *pentagonal fragment* is a fragment with only pentagonal inner faces. For a fragment  $B$ , all 2-degree vertices of  $B$  lie on its boundary. A path  $P$  on  $\partial(B)$  connecting two 2-degree vertices is *degree-saturated* if  $P$  contains no 2-degree vertices of  $B$  as intermediate vertices.

**Lemma 2.11.** ([23], Lemma 2.2) *Let  $B$  be a fragment of a fullerene graph  $F$  and  $W$  the vertex set consisting of all 2-degree vertices on  $\partial(B)$ . If  $0 < |W| \leq 4$ , then  $T = F - (V(B) \setminus W)$  is a forest and*

- (1)  $T$  is  $K_2$  if  $|W| = 2$ ;
- (2)  $T$  is  $K_{1,3}$  if  $|W| = 3$ ;
- (3)  $T$  is either the union of two  $K'_2$ 's, or a 3-length path, or  $T_0$  as shown in Fig. 8 if  $|W| = 4$ .

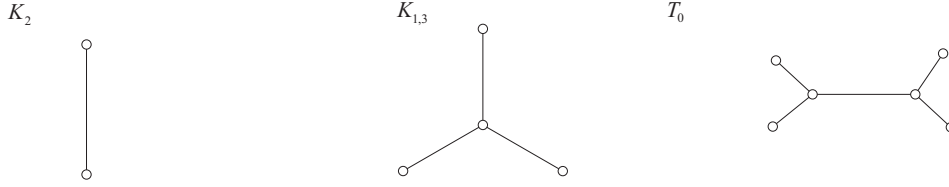


Figure 8. Trees:  $K_2$ ,  $K_{1,3}$  and  $T_0$ .

### 3 Proof of Theorem 1.1

Let  $F$  be a fullerene graph without  $L$  or  $R$  that is different from anyone of graphs in Fig. 2. To the contrary, suppose  $F$  is not 2-resonant. That is, there exist two disjoint hexagons  $h_1, h_2$  such that  $F - V(h_1 \cup h_2)$  does not have a perfect matching. Then Theorem 2.2 guarantees the existence of a vertex set  $A$  of  $F - V(h_1 \cup h_2)$  such that every component of  $F - (V(h_1 \cup h_2) \cup A)$  is factor-critical and the number of these factor-critical components is more than  $|A|$ . We make the following notations.

$$H = V(h_1 \cup h_2),$$

$\mathcal{D}$ : the collection of factor-critical components of  $F - H - A$ , and

$\mathcal{D}^*$ : the collection of non-trivial factor-critical components of  $F - H - A$ .

For convenience, let  $D$  and  $D^*$  denote respectively the union of vertex-sets of components in  $\mathcal{D}$  and  $\mathcal{D}^*$ , and  $D_0 = D - D^*$ ; let  $E(\mathcal{D}^*)$  denote the union of edge-sets of components in  $\mathcal{D}^*$ . Then  $D_0$  is an independent set of  $F$ . In Fig. 9 we divide  $V(F)$  into  $H, A, D^*$ ,  $D_0$ .

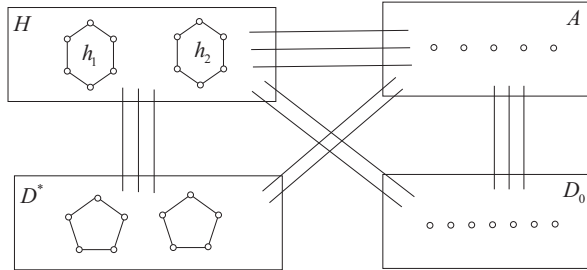


Figure 9. The partition of  $V(F)$  into  $H, A, D^*$  and  $D_0$ .

In what follows, we will show that the components in  $\mathcal{D}$  are singletons. That is,  $D^* = \emptyset$ , and the vertices of  $F$  can be reclassified into  $H, A$  and  $D_0$ . Finally, by means of the structures of the neighboring faces of  $h_1, h_2$  and the fact that  $F$  contains no subgraph  $L$  or  $R$  we can construct the fullerene graphs satisfying the conditions. In our constructing process we get the contradictions.

Since  $|\mathcal{D}|$  and  $|A|$  have the same parity and  $|\mathcal{D}| > |A|$ , we have

$$|\mathcal{D}| \geq |A| + 2. \quad (1)$$



Furthermore,  $A \cup H$  sends out exactly  $|\nabla(A \cup H)|$  edges, and

$$|\nabla(A \cup H)| = |\nabla(H)| + 3|A| - 2|E(A, H)| - 2|E(A, A)|. \quad (2)$$

Although there is no even components in  $\mathcal{D}$ , here we still use the notation system in the proof of Kaiser's [10]. Let

$$s(\mathcal{D}) = \sum_{F^* \in \mathcal{D}} \frac{|\nabla(F^*)| - 3}{2},$$

where each term is non-negative since  $F$  is 3-connected. Then  $D$  sends out precisely  $\nabla(D)$  edges, and

$$|\nabla(D)| = \sum_{F^* \in \mathcal{D}} |\nabla(F^*)| = 3|\mathcal{D}| + 2s(\mathcal{D}) \quad (3)$$

As  $|\nabla(A \cup H)| = |\nabla(D)|$ , (1), (2) and (3) imply that

$$|E(A, H)| + |E(A, A)| + s(\mathcal{D}) \leq \frac{1}{2}|\nabla(H)| - 3. \quad (4)$$

$|\nabla(H)|$  equals 12 or 10 by Lemma 2.9. From (4) we have

$$|E(A, H)| + |E(A, A)| + s(\mathcal{D}) \leq 3 \text{ if } h_1 \text{ and } h_2 \text{ are not incident, and} \quad (5)$$

$$|E(A, H)| + |E(A, A)| + s(\mathcal{D}) \leq 2 \text{ if } h_1 \text{ and } h_2 \text{ are incident.} \quad (6)$$

Let  $\mathcal{P}$  be the set of pentagons of  $F$ . Then  $|\mathcal{P}| = 12$ . If  $X \subseteq V(F)$  and  $Y \subseteq E(F)$ , let  $p(X) = |\{P \in \mathcal{P} | V(P) \cap X \neq \emptyset\}|$  and  $p(Y) = |\{P \in \mathcal{P} | E(P) \cap Y \neq \emptyset\}|$ . Observe that  $F - H - D^* - E(A, A)$  is a bipartite graph with bipartition  $D_0 \cup A$ . That is,

*all twelve pentagons of  $F$  must contain a vertex in  $H \cup D^*$  or an edge in  $E(A, A)$ .* (\*)

In particular,  $p(H) + p(E(A, A)) + p(D^*) \geq 12$  (7)

Since  $F$  contains no  $L, R$  as subgraphs,  $p(V(h_1)) \leq 4$ ,  $p(V(h_2)) \leq 4$ , and  $p(V(F^*)) \leq 3$  if  $F^* \in \mathcal{D}^*$  is a pentagon. (8)

**Lemma 3.1.**  *$F$  has no non-trivial cyclic 5-edge cuts.*

*Proof.* For a cyclic 5-edge-cut of  $F$ , it must be a *trivial* one; otherwise,  $F$  would be isomorphic to  $G_k$  for some integer  $k \geq 1$  by Theorem 2.3, which has the subgraph  $R$ , a contradiction.  $\square$

By Observation 2.1 and Lemma 3.1, we have

**Proposition 3.2.** *A component  $F^*$  in  $\mathcal{D}$  with  $|\nabla(F^*)| = 5$  is a pentagon.*

For convenience, in some of the following figures, the black vertices and the crossed vertices always represent the vertices belonging to  $A$  and  $D_0$  respectively, the non-trivial factor-critical components are drawn with black lines and the grey hexagons refer to  $h_1$  and  $h_2$ .

For a face  $f$  of  $F$ , we call an edge  $e$  on  $\partial(f)$  a *contributing edge* if it belongs to  $E(A, H)$ ,  $E(A, A)$ ,  $E(\mathcal{D}^*)$ , or  $E(h_1, h_2)$ . More precisely, edges in  $E(A, H)$ ,  $E(A, A)$ ,  $E(\mathcal{D}^*)$ ,  $E(h_1, h_2)$  are sometimes called the  $E(A, H)$ ,  $E(A, A)$ ,  $E(\mathcal{D}^*)$ ,  $E(h_1, h_2)$  edges, respectively. Lemma

2.9 implies  $|E(h_1, h_2)| \leq 1$ . By Ineqs. (5) and (6),  $|E(A, H)| \leq 3$ ,  $|E(A, A)| \leq 3$ , and  $s(\mathcal{D}) \leq 3$ . The latter implies  $|\nabla(F^*)| \leq 9$  for each  $F^* \in \mathcal{D}$ .

For  $F^* \in \mathcal{D}^*$ ,  $F^*$  is a 2-connected factor-critical graph that is a subgraph of  $F$ . If  $|\nabla(F^*)| \leq 7$ ,  $F^*$  has exactly one face that is not a face of  $F$  by Lemma 2.11. Hence  $F^*$  can be viewed as a fragment of  $F$ . If  $|\nabla(F^*)| = 9$ , by the similar reason together with  $c\lambda(F) = 5$  we have that  $F^*$  has at most two faces that are not faces of  $F$  and both  $h_1$  and  $h_2$  lie in the same such face of  $F^*$ . So we make a convention: both  $h_1$  and  $h_2$  lie in the exterior face of  $F^*$ .

We now give some characterizations about the faces in  $F$ .

**Lemma 3.3.** *For  $j \in \{1, 2\}$ , if a neighboring face  $f$  of  $h_j$  includes no contributing edges, then  $f$  is either a pentagon with the boundary  $HHD_0AD_0$  (which means the vertices on  $\partial(f)$  are consecutively in  $H, H, D_0, A, D_0$ ). The following notations have the same sense) or a hexagon with the boundary  $HHD_0HHD_0$  which is adjacent to both  $h_1$  and  $h_2$ .*

The lemma can be proved in the same way as Lemma 9 in [10].

**Corollary 3.4.** *At least one neighboring face of  $h_j$  contains a contributing edge for  $j \in \{1, 2\}$ .*

*Proof.* To the contrary, suppose every neighboring face of  $h_j$  has no contributing edges. By Lemma 3.3 the neighboring hexagonal faces of  $h_j$  intersect  $h_{3-j}$ . Using Lemma 2.8 we can know at most one neighboring face of  $h_j$  is hexagonal. Then its remaining neighboring pentagonal faces form a subgraph of  $F$  containing  $L$ , contradicting the assumption.  $\square$

Lemma 3.3 can be generalized as the following results.

**Lemma 3.5.** *For a neighboring face  $f$  of  $h_j$ ,  $j \in \{1, 2\}$ , we have the following results.*

(1) *If  $f$  contains precisely one contributing edge, which belongs to  $E(\mathcal{D}^*)$ ,  $E(A, H)$  and  $E(A, A)$ , respectively, then  $f$  is a hexagon with the boundaries  $HHD^*D^*AD_0$ ,  $HHAD_0AD_0$  and  $HHD_0AAD_0$ .*

(2) *If  $f$  contains precisely one contributing edge, which belongs to  $E(h_1, h_2)$ , then  $f$  is a pentagon with the boundary  $HHHHD_0$ .*

(3)  *$f$  cannot contain two  $E(A, A)$  edges.*

(4)  *$f$  cannot contain one  $E(A, A)$ , one  $E(\mathcal{D}^*)$  and one  $E(A, H)$  edge.*

(5)  *$f$  cannot contain two  $E(A, H)$  edges and one  $E(\mathcal{D}^*)$  edge.*

*Proof.* Let  $ab$  be a common edge of  $h_j$  and  $f$ ,  $a', b'$  the neighbors of  $a, b$ , respectively, not on  $h_j$ ,  $x$  the neighbor of  $a'$  on  $\partial(f)$  but different from  $a$  (see Fig. 10(a)).



Figure 10. (a) The labellings of face  $f$ , (b) Illustration to the proof of Lemma 3.5(4).

(1) If exactly one contributing edge of  $f$  belongs to  $E(\mathcal{D}^*)$ , then  $a', b' \in D^* \cup D_0$ . So one of  $a', b'$ , say  $a'$ , belongs to  $D^*$ . Otherwise, there would exist edges between  $D_0$  and  $D^*$ , a contradiction. Further  $x \in D^*$  by Observation 2.1. Now  $b' \in D_0$  as  $a'x \in E(\mathcal{D}^*)$  being a unique contributing edge of  $f$ . Moreover,  $b'$  and  $x$  must have a common neighbor belonging to  $A$  on  $\partial(f)$  since there are no edges between  $D_0$  and  $D^*$ . Thus  $f$  is a hexagon with the boundary  $HH D^* D^* A D_0$ . If  $f$  contains exactly one  $E(A, H)$  edge, then we have that  $a', b' \in A \cup D_0$  in an analogous way. Further, we have  $a' \in A$  and  $b' \in D_0$ , say. Also  $x \in D_0$ . Again  $b'$  and  $x$  must share a common neighbor belonging to  $A$  on  $\partial(f)$ . Then  $f$  is a hexagon with the boundary  $HH A D_0 A D_0$ . If  $f$  contains exactly one  $E(A, A)$  edge, then  $a', b' \in D_0$  and  $f$  is a hexagon with the boundary  $HH D_0 A A D_0$ .

(2) Since  $f$  contains precisely one contributing edge, that belongs to  $E(h_1, h_2)$ , one of  $a', b'$ , say  $a'$ , belongs to  $V(h_2)$ , the other  $b'$  to  $D_0$ . Then also  $x \in V(h_2)$  by the 3-regularity of  $F$ . Now  $b'$  and  $x$  must be adjacent since  $aa'$  is only one contributing edge of  $f$ . Hence  $f$  is a pentagon with the boundary  $HH H H D_0$ .

(3) Suppose  $f$  contains two  $E(A, A)$  edges  $e_1$  and  $e_2$ . Then  $f$  includes at least one  $E(A, H)$  edge. Ineqs. (5) and (6) imply that  $f$  is a hexagon,  $|E(A, H)| = 1$ ,  $|E(A, A)| = 2$ , and  $s(\mathcal{D}) = 0$ . The latter together with  $c\lambda(F) = 5$  imply that each component in  $\mathcal{D}$  is a single vertex; that is  $D^* = \emptyset$ . Hence  $p(H) + p(E(A, A)) + p(D^*) \leq (p(V(h_1)) + p(V(h_2))) + p(\{e_1, e_2\}) + 0 \leq (4 + 4) + 2 = 10$ , contradicting Ineq. (7).

(4) Suppose  $f$  contains one  $E(A, A)$ , one  $E(\mathcal{D}^*)$  and one  $E(A, H)$  edges. Then  $f$  is a hexagon with the boundary  $HH A A D^* D^*$ . Let  $aa'xyb'ba$  be the boundary of  $f$  along clockwise direction. Without loss of generality, we may assume  $a', x \in A, b', y \in D^*$ . Then by Ineqs. (5) and (6) we can know that  $|E(A, H)| = 1, |E(A, A)| = 1$  and  $s(\mathcal{D}) = 1$ . The latter together with Prop. 3.2 imply  $|\mathcal{D}^*| = 1$  and the component in  $\mathcal{D}^*$  is a pentagon. Let  $f_1$  and  $f_2$  be the common neighboring faces of  $h_j$  and  $f$ ,  $F^*$  the pentagonal component in  $\mathcal{D}^*$  (see Fig. 10(b)). Then both  $f_1$  and  $f_2$  are hexagons by (1). Moreover,  $p(D^*) = p(V(F^*)) \leq 3$  by Ineq. (8). So  $p(H) + p(E(A, A)) + p(D^*) \leq (p(V(h_1)) + p(V(h_2))) + p(\{a'x\}) + p(V(F^*)) \leq (4 + 3) + 1 + 3 = 11$ , contradicting Ineq. (7).

(5) Suppose  $f$  contains two  $E(A, H)$  edges and one  $E(\mathcal{D}^*)$  edge. In an analogous way as (4), we show that  $f$  is a hexagon and the component in  $\mathcal{D}^*$ , say  $F^*$ , is a pentagon. Moreover,  $p(E(A, A)) = 0, p(D^*) = p(V(F^*)) \leq 3$ . Again we have  $p(H) + p(E(A, A)) + p(D^*) \leq (4 + 4) + 3 = 11$ , a contradiction.  $\square$

Note: By Observation 2.1 and Lemma 3.5 (5) we can know if  $h_i$  is not incident to a non-trivial factor-critical component  $F^*$ , then the neighboring faces of  $h_i$  do not contain  $E(F^*)$  edges for  $i \in \{1, 2\}$ . This fact is used elsewhere in this paper.

**Observation 3.6.** *Let  $f$  be a face of  $F$  with precisely one contributing edge, which belongs to  $E(A, A)$ . Then  $f$  is either a pentagon with the boundary  $AAD_0AD_0$  or a hexagon with the boundary  $HHD_0AAD_0$ .*

*Proof.* Let  $ab \in E(A, A) \cap E(f)$  and  $a', b'$  be the neighbors of  $a, b$ , respectively, on  $\partial(f)$  but  $a' \neq b, b' \neq a$ . Then  $a', b' \in D_0$  as  $f$  contains precisely one contributing edge  $ab$ . If  $f$  is a pentagon, then  $a'$  and  $b'$  must have a common neighbor on  $\partial(f)$  belonging to  $A$  and  $f$  has the form  $AAD_0AD_0$ . If  $f$  is a hexagon  $aa'xyb'ba$ , then  $x, y \notin D$ . Since  $f$  has no other  $E(A, A)$  edge except for  $ab$ , one of  $x$  and  $y$  belongs to  $H$  and  $xy$  belongs to  $E(h_i)$  for some  $i \in \{1, 2\}$ . Thus  $f$  has the form  $HHD_0AAD_0$ .  $\square$

**Observation 3.7.** *Let  $f$  be a face of  $F$  containing no contributing edges and  $f \neq h_1, f \neq h_2$ . Then  $f$  is either a pentagon with the boundary  $HHD_0AD_0$  or a hexagon intersecting both  $h_1$  and  $h_2$  and with the boundary  $HHD_0HHD_0$  or a hexagon with the boundary  $AD_0AD_0AD_0$ .*

*Proof.* If  $f$  is a pentagon, then by (\*),  $f$  contains a vertex of  $H$ . Thus  $f$  is a neighboring face of  $h_i$  for some  $i \in \{1, 2\}$  by the 3-regularity of  $F$ . Hence  $f$  has the form  $HHD_0AD_0$  by Lemma 3.3. Now suppose  $f$  is a hexagon. Then  $V(f) \cap D^* = \emptyset$  as  $f$  contains no contributing edges. If  $f$  has a vertex of  $H$ , then  $f$  is a neighboring face of  $h_i$  for some  $i \in \{1, 2\}$ , without contributing edges. Thus  $f$  intersects both  $h_1$  and  $h_2$  with the boundary  $HHD_0HHD_0$  by Lemma 3.3. Otherwise all vertices of  $f$  belong to  $A \cup D_0$  and  $f$  has the form  $AD_0AD_0AD_0$ .  $\square$

We continue with some structural lemmas.

**Lemma 3.8.** *Let  $F^* \in \mathcal{D}^*$  be a pentagon denoted clockwise by  $v_1v_2v_3v_4v_5v_1$ . If  $f$  is a neighboring face of  $F^*$  with  $v_2v_2'xyv_1'v_1v_2$ , then one of the following assertions holds.*

(1)  $f$  is a hexagon, and (i)  $v_1', y \in V(h_i), v_2', x \in V(h_{3-i})$  for  $i \in \{1, 2\}$ , or (ii)  $v_1', y \in V(h_i), v_2' \in A, x \in D_0$  ( $v_2', x \in V(h_i), v_1' \in A, y \in D_0$ ), or (iii)  $v_1', v_2' \in A, x, y \in D^* \setminus V(F^*)$ , or (iv)  $v_1', v_2', x \in A, y \in D_0$  ( $v_1', v_2', y \in A, x \in D_0$ ).

(2)  $f$  is a pentagon ( $x = y$ ), and (v)  $v'_1, x \in V(h_i), v'_2 \in A$  ( $v'_2, x \in V(h_i), v'_1 \in A$ ) for  $i \in \{1, 2\}$ , or (vi)  $v'_1, v'_2 \in A, x \in D_0$ .

*Proof.* By Lemma 2.7,  $|E(f) \cap E(F^*)| = 1$ . That is,  $v'_1, v'_2, x, y \notin V(F^*)$ . Let  $f_1, f_2, f_3, f_4$  be the neighboring faces of  $F^*$  along the edges  $v_2v_3, v_3v_4, v_4v_5, v_5v_1$ , respectively. Denote by  $v'_3, v'_4, v'_5$  the neighbors of  $v_3, v_4, v_5$ , respectively, not in  $F^*$ .

We first consider the case that  $f$  is a hexagonal face. Since  $v'_1, v'_2 \in H \cup A$ , we have the following three cases.

*Case 1.*  $v'_1, v'_2 \in H$ . Let  $v'_1 \in V(h_i)$  and  $v'_2 \in V(h_j)$  for  $i, j \in \{1, 2\}$ . Then  $v'_1y \in E(h_i)$  and  $v'_2x \in E(h_j)$ . Lemma 2.7 implies that  $i \neq j$ . So (i) holds.

*Case 2.*  $v'_1 \in H, v'_2 \in A$  or  $v'_1 \in A, v'_2 \in H$ . By symmetry, we only need to consider the former situation. Then  $v'_1y \in E(h_i)$  for some  $i \in \{1, 2\}$  by the 3-regularity of  $F$ . Further by Observation 2.1, we have that  $x \notin H \cup D^*$  and  $x \in A \cup D_0$ . Then  $x \in D_0$  and (ii) holds. Otherwise,  $x \in A$ . Then  $f$  would contain three contributing edges  $v_1v_2, v'_2x, xy$  belonging to  $E(\mathcal{D}^*), E(A, A), E(A, H)$ , respectively, contradicting Lemma 3.5(4).

*Case 3.*  $v'_1, v'_2 \in A$ . Then  $x, y \notin H$ . Otherwise  $xy \in E(h_i)$  for some  $i \in \{1, 2\}$ . That is,  $f$  is a neighboring face of  $h_i$  with two  $E(A, H)$  edges and one  $E(\mathcal{D}^*)$  edge, contradicting Lemma 3.5(5). If  $x \in A$ , then  $y \notin D^*$  by Observation 2.1. Also  $y \notin A$ ; otherwise,  $|E(A, A)| \geq 3$  and  $s(\mathcal{D}) \geq 1$ , contradicting Ineq. (5). So  $y \in D_0$  and (iv) holds. If  $x \in D^* \setminus V(F^*)$ , then  $y \in D^* \setminus V(F^*)$  by Observation 2.1, and (iii) holds. If  $x \in D_0$ , then  $y \in H \cup A$ . Obviously,  $y \notin H$ . So  $y \in A$  and (iv) holds.

Now suppose  $f$  is a pentagon. Also,  $v'_1, v'_2 \in H \cup A$ . Further, either one of  $v'_1, v'_2$  belongs to  $H$ , the other to  $A$  or both  $v'_1, v'_2$  belong to  $A$ . If the former holds, then by symmetry, we may assume  $v'_1x \in E(h_i)$  for  $i \in \{1, 2\}$  and  $v'_2 \in A$ . Hence (v) holds. If the latter holds, then also  $x \notin H \cup D^*$ . We claim that  $x \in D_0$ , and (vi) holds. Suppose to the contrary that  $x \in A$ . Then  $f$  contains three contributing edges  $v_1v_2, v'_2x, v'_1x$ . Ineqs. (5) and (6) imply that  $E(A, H) = \emptyset$ ,  $E(A, A) = \{v'_2x, v'_1x\}$ , and  $s(\mathcal{D}) = 1$ . That is, there are no other contributing edges except for  $v'_2x, v'_1x$  and  $E(F^*)$ . Moreover, both  $f_1$  and  $f_4$  are hexagons by the assumption and do not contain contributing edges except for  $E(F^*)$ . This fact means  $v_3$  and  $v_5$  are incident to  $h_i$  and  $h_{3-i}$ , respectively, for  $i \in \{1, 2\}$  by applying Lemma 2.9 and Part (1) of this lemma to the faces  $f_1, f_4$ . Furthermore,  $f_1, f_2, f_3, f_4$  are hexagons with the boundaries  $D^*D^*HHD_0A$  by Lemma 3.5(1). Then we have the configuration shown in Fig. 11. Note at least one neighboring face of  $h_i$  different from  $f_1, f_2$  is hexagonal in order to prevent the forbidden subgraph  $L$  occurring in  $F$ . So  $p(V(h_i)) \leq 3$ . Similarly,  $p(V(h_{3-i})) \leq 3$ . Hence,  $p(H) + p(E(A, A)) + p(D^*) \leq (3 + 3) + 3 + 2 = 11$ , contradicting Ineq. (7). This contradiction verifies the claim.  $\square$

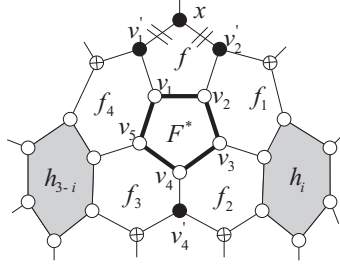


Figure 11. Illustration to the proof of Lemma 3.8:  $f$  is a pentagon.

Some extensions to Lemma 3.8 can be obtained in a similar way as follows.

**Lemma 3.9.** *Let  $F^* \in \mathcal{D}^*$  with a neighboring face  $f$ . Assume that  $f$  a path  $a'P(a,b)b'$  on its boundary such that  $P(a,b) \subset F^*$  and  $a', b' \notin V(F^*)$ .*

*If  $P(a,b)$  is of length one and  $f$  is a hexagon with  $a', b' \in A$ , then  $\partial(f)$  has the form  $D^*D^*AAD_0A$  or  $D^*D^*AD^*D^*A$ .*

*If  $f$  contains contributing edges belonging only to  $E(\mathcal{D}^*)$ , then  $a', b' \in A \cup H$  and one of the following assertions holds.*

- (i)  $a'b' \in E(h_i)$  for some  $i \in \{1, 2\}$ .
- (ii)  $a', b' \in A$  and  $a', b'$  share a common neighbor belonging to  $D_0$  or  $f$  is a hexagon with the boundary  $D^*D^*AD^*D^*A$ .
- (iii) one of  $a', b'$  belongs to  $A$ , the other to  $H$  and  $f$  is a hexagon with the boundary  $D^*D^*HHD_0A$ .

Regarding to the non-trivial factor-critical components of  $F - (H \cup A)$ , we have the following stronger conclusion.

**Lemma 3.10.** *Let  $F^* \in \mathcal{D}^*$  with  $|\nabla(F^*)| \geq 7$ . Then at least one of  $h_1, h_2$  is incident to  $F^*$ . In particular, if  $|\nabla(F^*)| = 9$ , then both  $h_1$  and  $h_2$  are incident to  $F^*$ .*

*Proof.* Suppose that  $h_i$  is not incident to  $F^*$  for some  $i \in \{1, 2\}$ . By the first part of Lemma 3.9 we have that the neighboring faces of  $h_i$  contain no  $E(F^*)$  edges. On the other hand, there must exist contributing edges contained in the neighboring faces of each of  $h_1$  and  $h_2$  by Corollary 3.4. So  $h_i$  has a neighboring face with a contributing edge not in  $E(F^*)$ .

If  $|\nabla(F^*)| = 9$ , Ineqs. (5) and (6) imply that  $F$  has only contributing  $E(F^*)$  edges. Hence both  $h_1$  and  $h_2$  have a neighboring face with  $E(F^*)$  edge. This contradiction shows that both  $h_1$  and  $h_2$  are incident to  $F^*$ .

In the following suppose  $|\nabla(F^*)| = 7$ . Suppose to the contrary that none of  $h_1$  and  $h_2$  are incident to  $F^*$ . Then  $s(\mathcal{D}) \geq s(F^*) = 2$ . Ineqs. (5) and (6) imply exactly one additional contributing edge in  $E(h_1, h_2)$ ,  $E(A, H)$ ,  $E(A, A)$ , or one  $F_1^* \in \mathcal{D}^*$  with  $|\nabla(F_1^*)| = 5$ . For

one  $E(h_1, h_2)$  edge existence, we can know  $h_1$  and  $h_2$  are incident and all of the neighboring faces of  $h_1$  and  $h_2$  are pentagons by Lemmas 3.3 and 3.5(2). Immediately the subgraph  $L$  occurs in  $F$ , contradicting the assumption. For one  $E(A, H)$  edge existence, the neighboring faces of  $h_i$  for some  $i \in \{1, 2\}$  don't contain contributing edges, contradicting Corollary 3.4. For one  $E(A, A)$  edge existence, the  $E(A, A)$  edge must be contained in one neighboring face of each of  $h_1$  and  $h_2$ , which is a hexagon by Lemma 3.5(1). Then the remaining five neighboring faces of  $h_i$  for  $i \in \{1, 2\}$  are either pentagons or hexagons intersecting both  $h_1$  and  $h_2$  by Lemma 3.3. So one of the remaining five neighboring faces of  $h_i$  must be a hexagon intersecting both  $h_1$  and  $h_2$  in order to avoid the occurrence of the forbidden subgraph  $L$ , which is impossible by Lemma 2.10. For  $F_1^*$  existence,  $F_1^*$  is a pentagon by Prop. 3.2, and both  $h_1$  and  $h_2$  are incident to  $F_1^*$  and four neighboring faces of  $F_1^*$  are hexagons with the boundaries  $D^*D^*HHD_0A$  by Lemma 3.5(1). By Lemma 3.3,  $h_1$  and  $h_2$  have a common neighboring face  $HHD_0HHD_0$  in order to prevent the occurrence of  $L$ , again contradicting Lemma 2.10.  $\square$

**Corollary 3.11.** *Let  $F^* \in \mathcal{D}^*$  with  $|\nabla(F^*)| = 7$ . Assume that  $F^*$  is not incident to  $h_i$  for some  $i \in \{1, 2\}$ . Then both  $h_i$  and  $h_{3-i}$  have a common neighboring face with the boundary  $HHD_0HHD_0$  and one of the remaining five neighboring faces of  $h_i$  contains either an  $E(A, V(h_i))$  edge such that  $p(V(h_i)) \leq 3$  or an  $E(A, A)$  edge with the boundary  $HHD_0AAD_0$  or an  $E(F_1^*)$  edge such that  $F_1^* \in \mathcal{D}^*$  is a pentagon and  $p(V(h_i)) \leq 3$ .*

*Proof.* From Lemma 3.10 and its proof we can know if  $h_i$  for  $i \in \{1, 2\}$  is not incident to  $F^*$  with  $|\nabla(F^*)| = 7$ , then  $h_{3-i}$  is incident to  $F^*$  and  $h_1$  and  $h_2$  are not incident. There is exactly one  $E(A, V(h_i))$  or  $E(A, A)$  or  $E(F_1^*)$  edge, contained in the neighboring faces of  $h_i$ . Note that an  $E(A, V(h_i))$  (or  $E(F_1^*)$ ) edge gives rise to two adjacent hexagonal neighboring faces of  $h_i$  by Lemma 3.5(1). For  $E(F_1^*)$  edge case,  $h_i$  is connected by exactly one edge by Lemma 2.9. Hence the other four neighboring faces of  $h_i$  have no contributing edges. Thus in order to prevent the occurring of the forbidden subgraph  $L$  one of the neighboring faces of  $h_i$  must be a hexagon  $HHD_0HHD_0$  intersecting both  $h_1$  and  $h_2$  by Lemma 3.3. For  $E(A, A)$  edge case, the proof is similar.  $\square$

By applying the above preliminary results we can obtain the following three critical lemmas. But their proofs will be presented in Sections 4, 5 and 6, respectively.

**Lemma 3.12.** *There is no  $F^* \in \mathcal{D}^*$  with  $|\nabla(F^*)| = 9$ .*

**Lemma 3.13.** *There is no  $F^* \in \mathcal{D}^*$  with  $|\nabla(F^*)| = 7$ .*

**Lemma 3.14.** *There is no  $F^* \in \mathcal{D}^*$  with  $|\nabla(F^*)| = 5$ .*

We will complete the proof by producing contradictions in all cases, where graph  $F_{42}$  in Fig. 2 is excluded. In fact the other ten graphs are excluded in proving Lemmas 3.12 to 3.14.

Before we have already shown that  $3 \leq |\nabla(F^*)| \leq 9$  for each  $F^* \in \mathcal{D}$ . Lemmas 3.12 to 3.14 imply that every component  $F^*$  in  $\mathcal{D}$  sends out exactly three edges. If  $F^*$  is non-trivial, then  $\nabla(F^*)$  forms a cyclic 3-edge-cut by Observation 2.1, contradicting that  $c\lambda(F) = 5$ . So the components in  $\mathcal{D}$  are singletons and we have the following claim.

**Claim 1.**  $D^* = \emptyset$ .

Then  $V(F) = H \cup A \cup D_0$  and  $F - H - E(A, A)$  is bipartite. So each pentagon of  $F$  must contain a vertex in  $H$  or an edge in  $E(A, A)$ . By (7) and (8) we have  $p(H) \leq 8$ , which means  $p(E(A, A)) \geq 4$ . On the other hand,  $p(E(A, A)) \leq 2|E(A, A)| \leq 6$  as  $|E(A, A)| \leq 3$  by Ineq. (5). So  $4 \leq p(E(A, A)) \leq 6$ .

**Claim 2.**  $p(E(A, A)) = 4$ .

*Proof.* If  $p(E(A, A)) \geq 5$ , then  $|E(A, A)| = 3$ , say  $e_1, e_2, e_3 \in E(A, A)$ . Moreover, at least two of  $e_1, e_2, e_3$ , say  $e_1, e_2$ , belong to two pentagons. In other words,  $e_1, e_2$  cannot be contained in the neighboring faces of  $h_1$  or  $h_2$  by Lemma 3.5(1) and (3). Hence the neighboring faces of  $h_i$  for some  $i \in \{1, 2\}$  do not include contributing edges as  $p(\{e_3\}) \geq 1$ , contradicting Corollary 3.4.  $\square$

**Claim 3.**  $|E(A, A)| = 3$ .

*Proof.* By Claim 2 we have  $|E(A, A)| \geq 2$ . If  $|E(A, A)| = 2$ , then each of these two  $E(A, A)$  edges belongs to two pentagons and none of these two  $E(A, A)$  edges is contained in the neighboring faces of  $h_j$  for  $j \in \{1, 2\}$  by Lemma 3.5(1) and (3). So there is exactly one other contributing edge belonging to  $E(A, H)$  or  $E(h_1, h_2)$  in  $F$  by Corollary 3.4 and Ineqs. (5) and (6). If this additional contributing edge belongs to  $E(A, H)$ , then it is in the neighboring faces of exactly one of  $h_1$  and  $h_2$ , contradicting Corollary 3.4. Otherwise,  $h_1$  and  $h_2$  are incident, we can obtain a subgraph  $L$  in  $F$  applying Lemmas 3.5(2) and 3.3 to the neighboring faces of  $h_1$  and  $h_2$  (In fact, we can obtain stronger result by Lemma 2.11: the neighboring faces of  $h_i$  are pentagons), contradicting the assumption.  $\square$

Put  $\{e_1, e_2, e_3\} = E(A, A)$ . Then we have the following result.

**Claim 4.** Each  $e_i$ ,  $i \in \{1, 2, 3\}$ , cannot be the intersection of two hexagonal faces of  $F$  for  $i \in \{1, 2, 3\}$ .

*Proof.* Suppose to the contrary that  $e_1$  belongs to two hexagons  $f_1$  and  $f_2$ . Then none of  $e_2$  and  $e_3$  can be contained in  $f_1$  or  $f_2$ ; Otherwise,  $p(E(A, A)) \leq 3$ , contradicting that



$p(E(A, A)) = 4$ . In other words, both  $f_1$  and  $f_2$  include only one  $E(A, A)$  edge. Then they are the neighboring faces of  $h_1$  and  $h_2$  by Observation 3.6. On the other hand, as  $p(E(A, A)) = 4$  and  $p(\{e_1\}) = 0, p(\{e_2\}) = p(\{e_3\}) = 2$ , which means the neighboring faces of  $h_1$  and  $h_2$  but different from  $f_1$  and  $f_2$  contain no contributing edges by Lemma 3.5(1) and (3). So in order to prevent the subgraph  $L$  occurring in  $F$ , by Lemma 3.3 we have that one of the neighboring faces of  $h_1$  must be a hexagon intersecting  $h_2$ , which is impossible by Lemma 2.10.  $\square$

**Claim 5.** Each pentagonal face  $f$  of  $F$  contains at most one of  $e_1, e_2, e_3$ .

*Proof.* Let  $uvwxyu$  be the pentagon  $f$ . If exactly two of  $e_1, e_2, e_3$  are contained in  $f$ , say  $uv, vw \in E(A, A)$ , then  $x, y \in D_0$ , contradicting that  $D_0$  is independent in  $F$ . If all of  $e_1, e_2, e_3$  are contained in  $f$ , say  $uv, vw, wx \in E(A, A)$  and  $y \in D_0$ , then the three neighboring faces of  $f$  (containing  $e_1, e_2, e_3$ , respectively) are pentagonal as  $p(E(A, A)) = 4$ . Then  $F$  includes  $R$ , a contradiction.  $\square$

From Claims 2 and 3, we can know there is precisely one  $E(A, A)$  edge (say  $e_3$ ) belonging to two pentagons. From Claim 4 each of  $e_1, e_2$  is the intersection of one pentagon and one hexagon. By Lemma 3.5(1),  $e_3$  cannot be contained in the neighboring faces of  $h_1$  and  $h_2$ . Since  $p(V(h_j)) = 4$ , at least two neighboring faces of  $h_j$  ( $j \in \{1, 2\}$ ) are hexagons. Thus in order to form the hexagonal neighboring faces of  $h_j$  and  $h_{3-j}$ ,  $e_1, e_2$  must be contained in the neighboring faces of  $h_j$  and  $h_{3-j}$  (respectively) with the boundaries  $HHD_0AAD_0$  and there exists a neighboring face of  $h_j$  intersecting both  $h_j$  and  $h_{3-j}$  with the boundary  $HHD_0HHD_0$ . More precisely, only two neighboring faces of  $h_j$  either containing the edge  $e_i$  or intersecting both  $h_j$  and  $h_{3-j}$  for  $i, j \in \{1, 2\}$  are hexagonal and the remaining four are pentagonal with the boundaries  $HHD_0AD_0$ . Hence the two hexagonal neighboring faces of  $h_j$  cannot be adjacent in order to prevent the subgraph  $L$  occurring in  $F$  for  $j \in \{1, 2\}$ . Denote by  $f_i(g_i)$  ( $1 \leq i \leq 6$ ) the neighboring faces of  $h_j(h_{3-j})$  in clockwise (anti-clockwise). Without loss of generality, suppose  $f_2 = g_2$  (see Fig. 12). By Lemma 2.10, the  $f_i$  and  $g_j$  are disjoint,  $4 \leq i, j \leq 6$ , and  $e_1$  belongs to some  $f_i$  and  $e_2$  to some  $g_j$  by the above analysis. So there are four cases for distributions of  $e_1, e_2$  on  $f_4, f_5, f_6$  and  $g_4, g_5, g_6$  by symmetry (see Fig. 12(a),(b),(c),(d)). Let  $f_7$  and  $f_8$  be two faces of  $F$  adjacent with  $f_3, f_4, g_3, g_4$  and  $f_1, f_6, g_1, g_6$ , respectively. If the case (a) or (b) or (c) holds, then a subgraph  $L$  consisting of  $f_3, f_4, g_3, g_4$  occurs, a contradiction. If the case (d) holds, then we have the configuration as shown in Fig. 12(d) by the above discussions, where  $f_7, f_8$  must be hexagons from Claim 5 as there is no  $R$  in  $F$  and  $v_1, v_2 \in D_0$ . Let  $G = h_j \cup h_{3-j} \bigcup_{i=1}^6 f_i \bigcup_{i=1}^6 g_i \cup f_7 \cup f_8$  be a fragment of  $F$ . Denote by  $f_9, \dots, f_{14}$  the neighboring faces of  $G$  as shown in Fig. 12(d). As  $f_1, g_1, g_6$

are pentagonal,  $f_9$  must be a hexagon by the assumption. So does  $f_{12}$  by symmetry. Hence  $v_3, v_4 \in A$  and  $v_5, v_6 \in D_0$  by Observation 3.7 (see Fig. 12(e)). Moreover,  $f_{11}$  and  $f_{14}$  are pentagons as  $p(\{e_1\}) = p(\{e_2\}) = 1$ . Thus  $v_7, v_8 \in D_0$ . Since  $D_0$  is independent in  $F$ ,  $v_6$  and  $v_8$  ( $v_5$  and  $v_7$ ) must share a common neighbor, say  $v_9$  ( $v_{10}$ ), belonging to  $A$  (see Fig. 12(e)). Finally  $v_9$  and  $v_{10}$  must be adjacent by Lemma 2.11 and we have the fullerene graph  $F_{42}$ , which is excluded in the assumption. Until now Theorem 1.1 is completed.

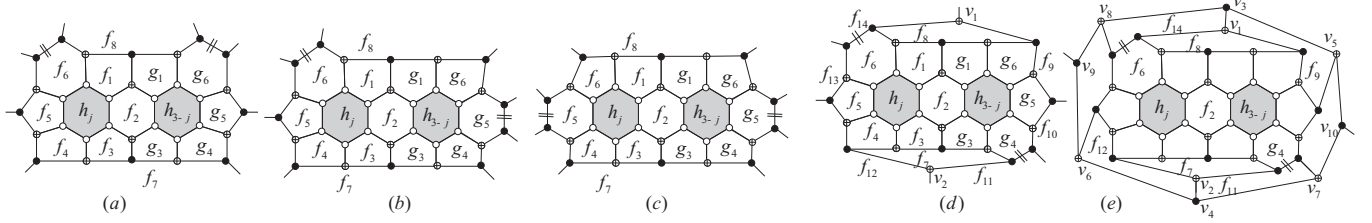


Figure 12. The four cases for distributions of  $e_1, e_2$  in  $f_4, f_5, f_6$  and  $g_4, g_5, g_6$ .

## 4 Proof of Lemma 3.12

Suppose to the contrary that  $F^* \in \mathcal{D}$  with  $|\nabla(F^*)| = 9$  exists. Then by Ineqs. (5) and (6),  $\mathcal{D}^*$  consists only of  $F^*$ ,  $E(A, A) = 0$  (so  $A$  is an independent set),  $E(A, H) = 0$  and  $|\nabla(H)| = 12$  (in particular,  $h_1$  and  $h_2$  are not incident). Thus both  $h_1$  and  $h_2$  are incident to  $F^*$  by Lemma 3.10. For  $j \in \{1, 2\}$ , let  $k = |\nabla(h_j) \cap \nabla(F^*)|$ . Obviously,  $1 \leq k \leq 6$ . Denote by  $v_1 v_2 v_3 v_4 v_5 v_6 v_1$  the boundary of  $h_j$  along the clockwise direction and  $f_1, f_2, \dots, f_6$  the six neighboring faces of  $h_j$  containing the edges  $v_1 v_2, v_2 v_3, \dots, v_6 v_1$ , respectively. Let  $G = h_j \cup \bigcup_{i=1}^6 f_i$ . Before our main argument, we give a definition for clusters.

A *cluster* at  $h_j$  ( $j = 1, 2$ ) is a sequence  $Q = (e_1, \dots, e_r)$  such that

- (1)  $e_i$  and  $e_{i+1}$  belong to the same face for each  $1 \leq i \leq r - 1$ ,
- (2)  $r \geq 3$  and for  $2 \leq i \leq r - 1$ ,  $e_i \in \nabla(F^*) \cap \nabla(h_j)$ , while  $e_1, e_r \in \nabla(F^*) \setminus \nabla(h_j)$ .

The size  $|Q|$  of  $Q$  is the number of its edges. Observe that  $e_1 \neq e_r$  by Lemma 2.7 and  $|Q| \geq 3$ .

For two edges  $e_1, e_2$  of  $F$ , we call  $e_1$  is *opposite*  $e_2$  if they both belong to the same hexagonal face of  $F$  and no edge of this boundary is incident with both  $e_1$  and  $e_2$ .

About the clusters we have the following properties.

**Claim 1.** Let  $j \in \{1, 2\}$ . Assume that the neighboring faces of  $h_j$  contain no other contributing edges except for  $E(\mathcal{D}^*)$ , then clusters at  $h_j$  are pairwise disjoint.

*Proof.* The proof of (1) is the same as Lemma 11 in [10]. We omit it here.  $\square$

**Claim 2.** There is at most one edge that is contained in a cluster  $Q_1$  at  $h_j$  and a cluster  $Q_2$  at  $h_{3-j}$  such that  $|Q_1| = 3, |Q_2| \geq 3$ .

*Proof.* Assume two edges  $e_1, e_2$  are contained in both  $Q_1$  and  $Q_2$ . The definition of a cluster and Lemma 3.9 guarantee  $e_1(e_2)$  cannot be contained in  $\nabla(h_1)$  or  $\nabla(h_2)$  and both  $e_1$  and  $e_2$  are opposite an edge of  $h_j$  and opposite an edge of  $h_{3-j}$ . If  $|Q_1| = 3$  and  $|Q_2| = 3$ , then by the above analysis and planarity of  $F$  we can obtain the configuration shown in Fig. 13(a). Immediately a quadrangular face occurs, contradicting the definition of  $F$ . If  $|Q_1| = 3$  and  $|Q_2| \geq 4$ , also a configuration depicted in Fig. 13(b) can be gained. However, in this case, a degree-saturated path of length more than six is obtained (see Fig. 13(b) the path along  $v_1, v_2, \dots, v_7$ ), contradicting that every face of  $F$  has a size of at most six.  $\square$

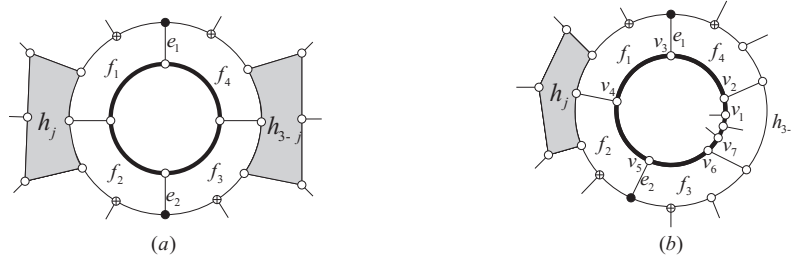


Figure 13. Clusters  $Q_1, Q_2$  at  $h_j$  and  $h_{3-j}$ : (a)  $|Q_1| = 3, |Q_2| = 3$ , (b)  $|Q_1| = 3, |Q_2| \geq 4$ .

Next we distinguish the following cases to complete the proof of Lemma 3.12.

*Case 1.*  $k = 1$ . Then there is exactly one cluster  $Q$  at  $h_j$  with  $|Q| = 3$ . Without loss of generality, suppose  $v_4$  is incident to  $F^*$  (see Fig. 14(a)). Then all of  $v_1, v_2, v_3, v_5, v_6$  are incident to  $D_0$  and  $f_3, f_4$  are hexagons with the boundaries  $D^*D^*HHD_0A$  by Lemma 3.9. In order to prevent the forbidden subgraph  $L$  occurring in  $F$ , at least one of  $f_1, f_2, f_5, f_6$  is hexagonal. Lemmas 3.3 and 2.8 imply exactly one of  $f_1, f_2, f_5, f_6$  must intersect both  $h_j$  and  $h_{3-j}$  with the boundary  $HHD_0HHD_0$  to form this hexagonal face.

If  $f_2$  intersects both  $h_j$  and  $h_{3-j}$ , then  $f_1, f_5$  and  $f_6$  are pentagons and  $a_i (1 \leq i \leq 5)$  belongs to  $A$  by Lemmas 3.3 and 3.9, where  $a_i (1 \leq i \leq 5)$  is shown in Fig. 14(a). Let  $w_1w_2w_3w_4w_5w_6w_1$  be the boundary of  $h_{3-j}$  along the anti-clockwise direction such that  $\partial(f_2) \cap \partial(h_{3-j}) = \{w_2w_3\}$ . Denote by  $f_7, f_8, \dots, f_{13}$  the seven neighboring faces of  $G$  as depicted in Fig. 14(a). If  $a_1$  is incident to  $F^*$ , then all of  $a_2, a_3, a_4$  and  $w_1$  are incident to  $F^*$  to form the neighboring faces  $f_7, f_8, f_9, f_{10}$  of  $F^*$  by Lemma 3.9 (see Fig. 14(b)). Now we obtain eight edges belonging to  $\nabla(F^*)$ . Thus  $w_4$  cannot be incident to  $F^*$ , otherwise,  $a_5$  is again incident to  $F^*$  to form the neighboring face  $f_{12}$  of  $F^*$  by Lemma 3.9 and  $|\nabla(F^*)| > 9$ , contradicting that  $|\nabla(F^*)| = 9$ . That is,  $w_4$  is incident to  $D_0$  and  $a_5$  and  $w_4$  share a common

neighbor, say  $d_1$  (see Fig. 14(b)). It's easy to see  $d_1$  cannot be again incident to  $h_{3-j}$ , which means  $d_1$  is again incident to a vertex belonging to  $A$ , say  $a_6$ . Furthermore,  $a_6$  must be incident to  $F^*$  to form the neighboring face  $f_{13}$  of  $F^*$  by Lemma 3.9. Then we obtain all the nine edges in  $\nabla(F^*)$  (see Fig. 14(b)) and three 2-degree vertices  $w_6, w_5$  and  $a_6$  that should have a common neighbor by Lemma 2.11. Immediately a triangular face occurs, which is impossible. So  $a_1$  is incident to  $D_0$  third times and shares a neighbor with  $w_1$ . Moreover,  $a_2, a_3, a_4$  are also incident to  $D_0$  third times as there are no edges between  $D_0$  and  $F^*$  and all of  $f_8, f_9, f_{10}$  are hexagons with the boundaries  $AD_0AD_0AD_0$  by Observation 3.7. Let  $a_6, a_7, a_8, d_1$  be the vertices shown in Fig. 14(c). If  $a_6$  is incident to  $D_0$  third times, then the neighboring face of  $h_{3-j}$  (say  $f_{14}$ ) including the edge  $w_1w_6$  is a pentagon by Lemma 3.3. Thus  $f_{14} \cup f_7 \cup f_1 \cup f_6$  forms a subgraph  $L$  in  $F$  (see Fig. 14(c)), contradicting the assumption. So  $a_6$  is incident to  $F^*$ . Similarly,  $w_6, a_7, a_8$  are also incident to  $F^*$ . By the planarity of  $F$ ,  $d_1$  is not incident to  $h_{3-j}$ . That is,  $d_1$  is incident to  $A$  third times. Let  $a_9$  be the third neighbor of  $d_1$ . Then  $a_9$  is incident to  $F^*$  with two edges to form two neighboring faces of  $F^*$  also by Lemma 3.9 (see Fig. 14(d)). We once again obtain nine edges in  $\nabla(F^*)$  and three 2-degree vertices  $w_5, w_4, a_5$  that must share a common neighbor. Also a triangular face occurs, a contradiction.

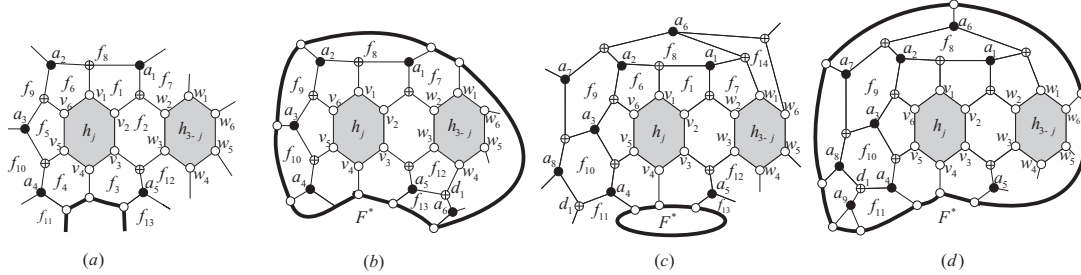


Figure 14. Illustration for Case 1 in the proof of Lemma 3.12.

So  $f_2$  cannot intersect both  $h_j$  and  $h_{3-j}$ . Thus it is a pentagon by Lemma 3.3. By symmetry,  $f_5$  is also a pentagon. If  $f_1$  intersects both  $h_j$  and  $h_{3-j}$ , then similarly as the case above, we always obtain nine edges in  $\nabla(F^*)$  and three 2-degree vertices that must share a common neighbor and finally occurs a triangular face, which is impossible.

Summarizing the above analysis, both  $h_1$  and  $h_2$  must be incident to  $F^*$  with at least two edges, that is,  $k \geq 2$ .

*Case 2.*  $k = 2$ . Then there are at most two clusters at  $h_j$ . If exactly one cluster  $Q_1$  exists at  $h_j$ . The definition of a cluster implies  $|Q_1| = 4$ . Without loss of generality, we may assume  $v_3, v_4$  are incident to  $F^*$ . Analogously as Case 1,  $f_2, f_4$  are hexagons with the boundaries  $HHD^*D^*AD_0$  and  $f_1, f_5, f_6$  are either pentagons with the boundaries  $HHD_0AD_0$

or hexagons intersecting both  $h_j$  and  $h_{3-j}$  with the boundary  $HHD_0HHD_0$ . For  $f_i$  ( $i = 1, 5, 6$ ) intersecting both  $h_j$  and  $h_{3-j}$ , we always obtain a triangular face in  $F^*$ , which is impossible. For  $f_1, f_5, f_6$  being pentagonal, we have the configuration shown in Fig. 15(a). Let  $a_i$  and  $f_j$  be shown in Fig. 15(a) for  $i \in \{1, \dots, 5\}, j \in \{7, \dots, 12\}$ . Then  $a_i \in A$ . If one of  $a_1, \dots, a_5$  is incident to  $F^*$ , then the remaining four are also incident to  $F^*$  and we have nine edges in  $\nabla(F^*)$ , but now  $h_1$  and  $h_2$  are missed, which is impossible. So all of  $a_1, \dots, a_5$  are incident to  $D_0$  third times and again  $f_8, f_9, f_{10}, f_{11}$  are hexagons with the boundaries  $AD_0AD_0AD_0$ . Let  $a_6, \dots, a_9, d_1, d_2$  and  $f_{13}, \dots, f_{17}$  be the vertices and faces of  $F$  as depicted in Fig. 15(b). If  $d_1$  is incident to  $h_{3-j}$ , then  $f_{12}$  is a hexagon with the boundary  $D^*D^*HHD_0A$  by Lemma 3.9 and all of  $a_6, \dots, a_9$  are incident to  $D_0$  third times, otherwise,  $|\nabla(F^*)| > 9$  (impossible). That is,  $f_{13}$  is a pentagon with the boundary  $HHD_0AD_0$  and  $f_{14}, f_{15}, f_{16}$  are hexagons with the boundaries  $AD_0AD_0AD_0$  (see Fig. 15(c)). Moreover,  $d_2$  can not be incident to  $h_{3-j}$  by the planarity of  $F$ . That is,  $d_2$  is incident to  $A$  third times, say  $a_{10} \in A$  (see Fig. 15(c)). Again  $a_{10}$  is incident to  $F^*$  to form the neighboring face  $f_7$  of  $F^*$  and  $f_{17}$  is also a hexagon with the boundary  $AD_0AD_0AD_0$ . Now we have six edges in  $\nabla(F^*)$  (see Fig. 15(c)). Let  $a_{11}, \dots, a_{14}, d_3$  be shown in Fig. 15(c). Apply the same analysis to  $a_{11}, \dots, a_{14}, d_3$  as  $a_6, \dots, a_9, d_2$  and repeat this procedure, finally we can gain nine edges in  $\nabla(F^*)$  but only one of them belongs to  $\nabla(h_{3-j})$ , contradicting that  $|\nabla(h_{3-j}) \cap \nabla(F^*)| \geq 2$ . This contradiction means  $d_1$  is incident to  $A$  third times, say  $a_{10}$ . Similarly for  $d_2$  (see Fig. 15(d)). Moreover,  $d_2$  can not be incident to  $a_{10}$  by the planarity of  $F$  and Lemma 2.11. Now we have a similar situation as the vertices  $a_1, \dots, a_5$  and we can repeat the above procedure to  $a_6, \dots, a_{11}$  until we gain nine edges in  $\nabla(F^*)$ , but  $h_{3-j}$  isn't incident to  $F^*$ , contradicting Corollary 3.4.

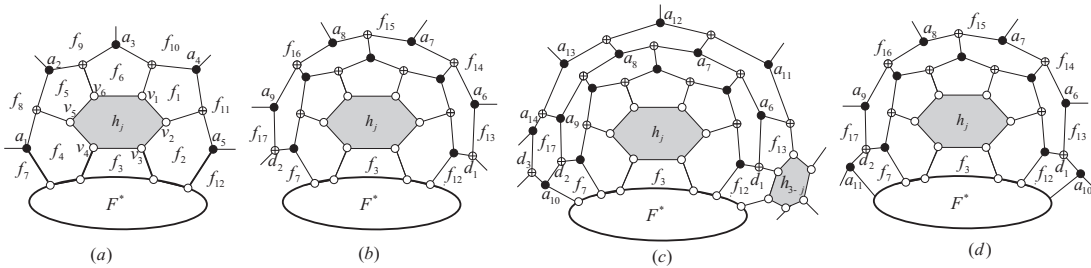


Figure 15. Illustration for Case 2 in the proof of Lemma 3.12.

If there are two clusters  $Q_1, Q_2$  at  $h_j$ . As  $|\nabla(h_j) \cap \nabla(F^*)| = 2$ , the definition of a cluster and Claim 1 imply  $|Q_1| = 3, |Q_2| = 3$  and we have obtained six edges in  $\nabla(F^*)$ . Thus there are at most three edges of  $\nabla(F^*)$  left for the clusters at  $h_{3-j}$ . Note that there are at most two edges belonging to the intersection of two clusters at  $h_j$  and  $h_{3-j}$ , respectively, and these

two edges cannot be in  $\nabla(h_j)$  or  $\nabla(h_{3-j})$ . So at most two clusters exist at  $h_{3-j}$ . Moreover, the former situation shows that there can not be one cluster at  $h_{3-j}$  of size 4. Summarizing the above analysis, we can know either precisely one cluster, say  $Q_3$ , exists at  $h_{3-j}$  satisfying  $|Q_3| = 5$  and  $|Q_3 \cap Q_i| = 2$  for some  $i \in \{1, 2\}$  or two clusters, say  $Q_3, Q_4$ , exist at  $h_{3-j}$  such that  $|Q_3| = |Q_4| = 3$  and  $|Q_i \cap Q_3| = 2$  or  $|Q_{3-i} \cap Q_4| = 2$  for some  $i \in \{1, 2\}$  or  $|Q_3| = 3, |Q_4| = 4$  and  $|Q_i \cap Q_3| = 2, |Q_{3-i} \cap Q_4| = 2$  for  $i \in \{1, 2\}$ . However, no matter which case happens will contradict Claim 2. So  $k \neq 2$ . In other words,  $|\nabla(h_i) \cap \nabla(F^*)| \geq 3$  for all  $i \in \{1, 2\}$ .

*Case 3.  $k \geq 3$ .* Let  $k = 3$ . Then also at most two clusters at  $h_j$  are obtained by Claim 1 and the fact that  $|\nabla(h_{3-j}) \cap \nabla(F^*)| \geq 3$ . As before, for exactly one cluster at  $h_j$  existence, we can always obtain a triangular face in  $F$  by checking for  $f_i$  ( $1 \leq i \leq 6$ ) intersecting both  $h_j$  and  $h_{3-j}$  or not (Note in this case  $|\nabla(h_{3-j}) \cap \nabla(F^*)| \geq 3$ ). For two clusters  $Q_1, Q_2$  at  $h_j$  existence, we have  $|Q_k| = 3$  and  $|Q_{3-k}| = 4$  for  $k = 1, 2$  and  $Q_k$  and  $Q_{3-k}$  are disjoint by Claim 1. That is, there are at most two edges of  $\nabla(F^*)$  left for the clusters at  $h_{3-j}$ , contradicting the fact that  $|\nabla(h_{3-j}) \cap \nabla(F^*)| \geq 3$ . So  $k \geq 4$ . However, in this case we once again obtain that  $|\nabla(F^*)| > 9$ , also a contradiction.

Summarizing the above discussion, such a  $F^*$  cannot exist in  $F$ .

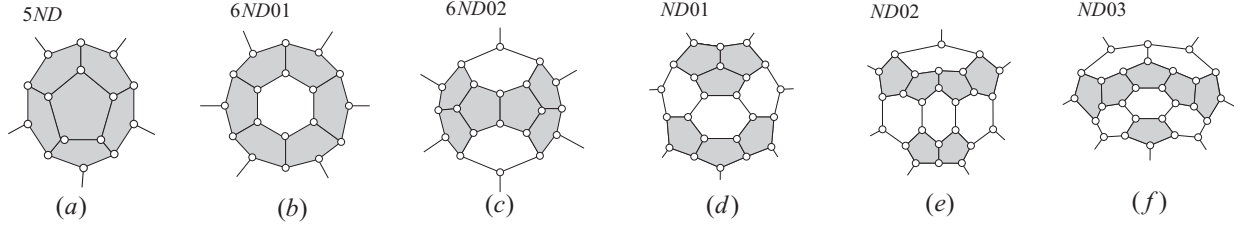
## 5 Proof of Lemma 3.13

Suppose to the contrary that  $F^* \in \mathcal{D}$  with  $|\nabla(F^*)| = 7$  exists. Then  $\nabla(F^*)$  forms a cyclic 7-edge-cut. If  $\nabla(F^*)$  is a degenerate cyclic 7-edge-cut, then  $F^*$  or  $\overline{F^*}$  contains less than six pentagons, which means  $F^*$  or  $\overline{F^*}$  is isomorphic to one component of  $D01, \dots, D57$  as shown in Fig. 5. However,  $F^*$  or  $\overline{F^*}$  is only possibly isomorphic to the components of  $D01, \dots, D09, D11$  as  $F$  can not possess  $L$  or  $R$  as subgraphs. Moreover, since  $F^*$  is 2-connected and  $\overline{F^*}$  contains two disjoint hexagons  $h_1$  and  $h_2$ ,  $F^*$  must be isomorphic to one component of  $D05, D08, D09, D11$  depicted in Fig. 5.

If  $\nabla(F^*)$  is a non-degenerate cyclic 7-edge-cut, then it can be constructed from the trivial ones using the reverse operations of  $(O_1), (O_2), (O_3)$  by Theorem 2.6. Note that there cannot be the subgraphs  $L$  or  $R$  in  $F$ . So in our construction process we stop extending the cyclic 7-edge-cuts as long as we encounter the two subgraphs. Denote by  $5D$  the trivial cyclic 5-edge-cut. In the following table we list the configurations that arise when applying operations  $(O_1^{-1}), (O_2^{-1})$  and  $(O_3^{-1})$  and in Fig. 16 we give the corresponding non-degenerated cyclic 7-edge-cuts.

Table 1: Generating the non-degenerated cyclic 7-edge-cut.

cut	5D	6D01	6D02	6D03	D01	D02	D03	D04	D05	D06	D07	D08	D09	D11
$O_1^{-1}$	6D02	D01	D02 D03 D04	D06 D07	—	—	—	—	—	—	—	—	—	—
$O_2^{-1}$	—	—	6D03	6D04	D05	D05 D06	D05 D06	D06 D07	D08	D08 D09 D10	D09 D10	D11 D12	D11 D13	D16 D17
$O_3^{-1}$	5ND	6ND01	—	6ND02	—	—	—	—	ND01	—	—	ND02	—	ND03


 Figure 16. The non-degenerated cyclic 7-edge-cuts when applying operations  $(O_1^{-1})$ ,  $(O_2^{-1})$  and  $(O_3^{-1})$ .

Since the components of  $5ND$ ,  $6ND01$ ,  $6ND02$ ,  $ND01$  and  $ND02$  (see Fig. 16 (a),(b),(c), (d),(e)) contract the assumption, combining this with the previous analysis, the components of  $\nabla(F^*)$  contain a subgraph isomorphic to the component of  $ND03$  depicted in Fig. 16(f).

By the above analysis, we only need to show the following three Claims hold in order to prove Lemma 3.13.

**Claim 1:**  $F^*$  can not be isomorphic to the component of  $D05$  as illustrated in Fig. 5.

*Proof.* To the contrary,  $F^*$  is the component of  $D05$ . Let  $v_1, \dots, v_7$  and  $f_1, \dots, f_7$  be the vertices and neighboring faces of  $F^*$  as shown in Fig. 17(a). Then  $v_1, \dots, v_7$  are incident to  $H$  or  $A$ . Moreover, there exists at most one  $E(h_1, h_2)$  or  $E(A, A)$  or  $E(A, H)$  edge or one pentagonal factor-critical component by Ineqs. (5) and (6) and Prop. 3.2. For convenience, we usually don't distinguish  $h_1$  and  $h_2$ .

By Lemma 3.10, at least one of  $v_1, \dots, v_7$  is incident to  $h_1$  or  $h_2$ . Firstly suppose  $v_1$  is incident to  $h_1$ . Then all of  $v_2, \dots, v_7$  can not be incident to  $h_1$  by Lemma 2.11. Let  $f_8, f_9, f_{10}, f_{11}$  be the four neighboring faces of  $h_1$  different from  $f_1, f_7$  (see Fig. 17(b)). Then  $v_2$  can not be incident to  $h_2$ , otherwise,  $h_1$  and  $h_2$  are incident and all of  $f_8, f_9, f_{10}, f_{11}$  are pentagons by Lemmas 3.3 and 3.5(2), thus a subgraph  $L$  occurs in  $F$  (impossible) (see Fig. 17(b)). So  $v_2$  is incident to  $A$ . By symmetry,  $v_7$  is also incident to  $A$ . If  $v_3$  is incident to  $h_2$ , then  $f_2$  contains an  $E(A, H)$  edge and there are no other contributing edges

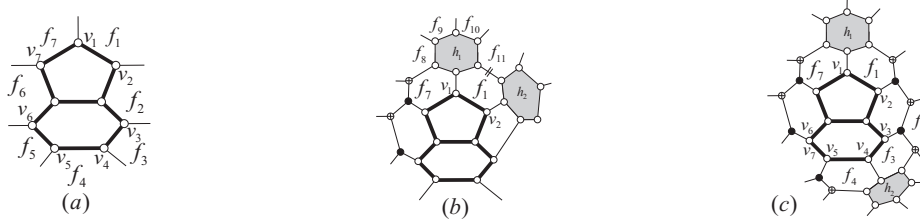


Figure 17. (a) The labellings of  $F^*$ , (b) the case  $v_1, v_2$  incident to  $h_1, h_2$  (respectively), and (c) the case  $v_1, v_4$  incident to  $h_1, h_2$  (respectively).

except for this  $E(A, H)$  edge and  $E(F^*)$ . Now applying Lemma 3.9 to  $f_1, \dots, f_7$  we have  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (4 + 4) + 3 = 11$ , contradicting Ineq. (7). So  $v_3$  is incident to  $A$ . Similarly for  $v_6$ . Now we may assume  $v_4$  is incident to  $h_2$ . Then all of  $f_1, f_3, f_4, f_7$  are hexagons with the boundaries  $D^*D^*HHD_0A$  (see Fig. 17(c)), otherwise,  $f_1$  or  $f_3$  or  $f_4$  or  $f_7$  contains an  $E(A, H)$  edge and by Lemma 3.9  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (4 + 4) + 3 = 11$  (impossible). Moreover,  $p(V(h_1)) \leq 3, p(V(h_2)) \leq 3$ . At this time at most one  $E(A, H)$  or  $E(A, A)$  edge or one pentagonal factor-critical component  $F_1^*$  exists in  $F$ . However, since an  $E(A, A)$  edge gives rise to at most two pentagons and  $p(V(F_1^*)) \leq 3$ , we obtain  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (3 + 3) + 2 + 3 = 11$  no matter which case occurs, also impossible by Ineq. (7). So  $v_4$  is incident to  $A$ . So does  $v_5$  by symmetry. Now applying Corollary 3.11 we can know  $f_{10}$  or  $f_{11}$  intersects both  $h_1$  and  $h_2$  with the boundary  $HHD_0HHD_0$ . Moreover, if there is an  $E(A, V(h_2))$  edge, then  $f_1, \dots, f_7$  contain no contributing edges. Thus  $f_1, f_2, f_6, f_7$  are hexagons and  $f_3, f_4, f_5$  are pentagons by Lemma 3.9. Hence  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (3 + 3) + 4 = 10$  (impossible). If there is an  $E(A, A)$  edge, say  $e \in E(A, A)$ , then for  $f_{11}$  intersecting both  $h_1$  and  $h_2$  we have all of  $f_8, f_9, f_{10}$  are pentagons with the boundaries  $HHD_0AD_0$  by Lemma 3.3 and  $e$  must be contained in  $f_{15}$  (see Fig. 18(a) for the labellings of  $f_{12}, \dots, f_{16}$ ), otherwise,  $f_{12} \cup f_{13} \cup f_{14} \cup f_{15}$  forms a subgraph  $L$  when  $e$  belongs to  $f_{16}$  and  $f_9 \cup f_{10} \cup f_{16} \cup f_{15}$  forms a subgraph  $L$  when  $e$  belongs to  $f_{12}$  or  $f_{13}$  or  $f_{14}$ . Now all of  $f_{12}, f_{13}, f_{14}$  are pentagons with the boundaries  $HHD_0AD_0$  by Lemma 3.3 and  $f_2$  is a hexagon with the boundary  $D^*D^*D^*AD_0A$  and  $f_3$  is a pentagon with the boundary  $D^*D^*AD_0A$  by Lemma 3.9. Immediately the vertex  $v$  (see Fig. 18(b)) belongs to three pentagons, contradicting the assumption. Similarly for  $f_{10}$  intersecting both  $h_1$  and  $h_2$  we have  $f_8, f_9, f_{11}$  are pentagons with the boundaries  $HHD_0AD_0$  and  $e$  must belong to  $f_{15}$ . Again we have a vertex  $v$  belongs to three pentagons (see Fig. 18(c)) (impossible). If there is a pentagonal factor-critical component  $F_1^*$ , then  $F_1^*$  can not be incident to  $h_1$ , otherwise,  $p(V(h_1)) \leq 1, p(V(F_1^*)) \leq 2$  and  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (3 + 1) + (4 + 2) = 10$  (impossible). Moreover,  $f_3$  or  $f_4$  or  $f_5$  must contain an  $E(F_1^*)$  edge to form a hexagonal neighboring face of  $F_1^*$  by Lemma



3.8 and the fact that  $p(V(F_1^*)) \leq 3$ . If  $f_3$  or  $f_5$  contains an  $E(F_1^*)$  edge, say  $f_3$ , then  $v_8$  is incident to  $h_2$  (see Fig. 18(d) for the labellings of  $v_8$  and the neighboring faces of  $F_1^*$ ) since  $F$  can not possess  $R$  as subgraph. Now by Lemmas 3.8 and 3.9 all of  $f_{12}, f_{15}, f_4, f_5$  are pentagons with the boundaries  $D^*D^*AD_0A$ . Thus  $f_{12} \cup F_1^* \cup f_{15} \cup f_4$  forms a subgraph  $L$  (see Fig. 18(d)), contradicting the assumption. If  $f_4$  contains an  $E(F_1^*)$  edge, then analogously  $f_{12} \cup F_1^* \cup f_{15} \cup f_5$  forms a subgraph  $L$  (see Fig. 18(e)) (also impossible). This contradiction means  $v_1$  can not be incident to  $h_1$ . Thus  $v_1$  is incident to  $A$ .

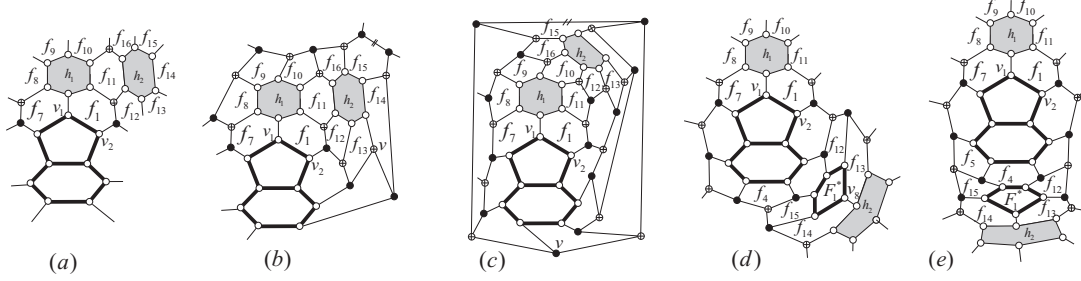


Figure 18. The case  $v_1$  incident to  $h_1$  and all of  $v_2, \dots, v_7$  incident to  $A$ .

Now suppose  $v_2$  is incident to  $h_1$ , then  $v_3$  is also incident to  $h_1$  to form the pentagon  $f_2$  (see Fig. 19(a)), otherwise,  $f_2$  includes an  $E(A, H)$  edge and there are no other contributing edges by Ineqs. (5) and (6), which means  $h_2$  is incident to  $F^*$  by Corollary 3.4. However, for  $v_4$  or  $v_5$  or  $v_6$  or  $v_7$  incident to  $h_2$  we'll obtain  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (4+4) + 3 = 11$  by Lemma 3.9 (impossible). On the other hand,  $f_1$  is a hexagon with the boundary  $D^*D^*HHD_0A$  since  $v_2$  can not belong to three pentagons by the assumption (see Fig. 19(a)). Then if  $v_4$  is incident to  $h_2$ , then  $f_3$  contains an  $E(h_1, h_2)$  edge (see Fig. 19(a)) and there are no other contributing edges, which means the neighboring faces of  $h_2$  different from  $f_3, f_4$  are pentagons and they form a subgraph  $L$  in  $F$  (impossible). So  $v_4$  is incident to  $A$ . If  $v_5$  is incident to  $h_2$ , then all of  $f_3, f_4, f_5$  are hexagons with the boundaries  $D^*D^*HHD_0A$  (see Fig. 19(b)), otherwise,  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (4+4) + 3 = 11$  (impossible). Let  $f_8, f_9, f_{10}$  and  $f_{11}, f_{12}, f_{13}, f_{14}$  be the neighboring faces of  $h_1$  and  $h_2$ , respectively (see Fig. 19(b)). Then at least one of  $f_{11}, f_{12}, f_{13}, f_{14}$  is hexagonal in order to avoid the occurrence of the subgraph  $L$ . On the other hand,  $f_8, \dots, f_{14}$  are pairwise different by Lemma 2.11. Thus at least one of  $f_{11}, f_{12}, f_{13}, f_{14}$  contains a contributing edge by Lemma 3.3. For an  $E(h_1, h_2)$  edge contained in  $f_j$  ( $11 \leq j \leq 14$ ),  $f_{11} \cup f_{12} \cup f_{13} \cup f_{14}$  forms a subgraph  $L$  by Lemma 3.5(2) (impossible). For an  $E(A, H)$  edge contained in  $f_j$  ( $11 \leq j \leq 14$ ), we have  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (2+4) + 3 = 9$  (impossible). For an  $E(A, A)$  edge contained in  $f_j$  ( $11 \leq j \leq 14$ ), we have  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (3+4) + 1 + 3 = 11$  (impossible). For an  $E(F_1^*)$  edge contained in  $f_j$  ( $11 \leq j \leq 14$ ) we have  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (2+2) + (3+2) = 9$  (impossible).

when  $F_1^*$  is incident to  $h_1$  and  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (2+4) + (3+3) = 12$  when  $F_1^*$  is not incident to  $h_1$ , but now  $f_2 \in \{P|V(P) \cap V(F^*) \neq \emptyset\} \cap \{P|V(P) \cap H \neq \emptyset\}$ , contradicting that  $|\mathcal{P}| = 12$ . So  $v_5$  is incident to  $A$ . If  $v_6$  is incident to  $h_2$ , then similarly as the cases above,  $v_7$  is also incident to  $h_2$  to form the pentagonal face  $f_6$  and  $f_1, f_3, f_5, f_7$  are all hexagons with the boundaries  $D^*D^*HHD_0A$  (see Fig. 19(c)). Now for none or an  $E(A, H)$  edge existence, we have  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (4+4) + 2 = 10$  (impossible). For an  $E(A, A)$  edge existence, say  $e \in E(A, A)$ , then  $e$  can not be contained in the neighboring faces of  $h_1$  or  $h_2$  or  $F^*$ , otherwise,  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (4+4) + 2 + 2 - 1 = 11$  (impossible). Thus the neighboring faces of  $h_1$  ( $h_2$ ) different from  $f_1, f_2, f_3$  ( $f_5, f_6, f_7$ ) are pentagons with the boundaries  $HHD_0AD_0$  and  $f_4$  is also a pentagon with the boundary  $D^*D^*AD_0A$  by Lemmas 3.3 and 3.9. Finally we obtain the fullerene graph  $F_{48}^1$  as shown in Fig. 19(d). For an  $E(F_1^*)$  edge existence, we have a subgraph  $R$  when  $F_1^*$  is neither incident to  $h_1$  nor to  $h_2$  (impossible) and the number of pentagons is at most  $(4+2)+(3+1)=10$  when  $F_1^*$  is incident to at least one of  $h_1, h_2$  by Lemmas 3.8, 3.9, contradicting that  $|\mathcal{P}| = 12$ . This contradiction means  $v_6$  is incident to  $A$ . If  $v_7$  is incident to  $h_2$ , then  $f_6$  contains an  $E(A, H)$  edge and  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (4+3) + 3 = 10$  (impossible). Thus  $v_7$  is also incident to  $A$ . Let  $f_8, f_9, f_{10}$  be the neighboring faces of  $h_1$  different from  $f_1, f_2, f_3$  (see Fig. 20(a)). Now we have a similar situation as the case  $v_1$  incident to  $h_1$  and for an  $E(A, A)$  edge existence, we have the fullerene graphs  $F_{44}^1, F_{44}^2$  as shown in Fig. 20(b),(c). For  $h_2$  incident to a pentagonal factor-critical component  $F_1^*$  we have the fullerene graph  $F_{46}^1$  as depicted in Fig. 20(d). But all of  $F_{44}^1, F_{44}^2$  and  $F_{46}^1$  are excluded in the assumption. Hence in the following we may assume  $v_2$  is incident to  $A$ . So does  $v_7$  by symmetry.

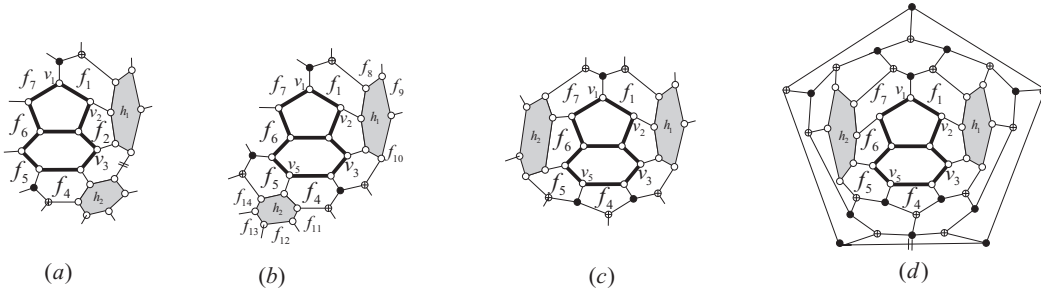


Figure 19. The case  $v_2, v_3$  incident to  $h_1$  and  $v_4$  or  $v_5$  or  $v_6, v_7$  incident to  $h_2$ .

Next suppose  $v_3$  is incident to  $h_1$ , then  $f_2$  contains an  $E(A, H)$  edge and  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (4+3) + 2 = 9$  (impossible). Thus  $v_3$  is also incident to  $A$ . So does  $v_6$  by symmetry. If  $v_4$  is incident to  $h_1$ , then  $v_5$  can not be incident to  $h_2$ , otherwise,  $h_1$  and  $h_2$  are incident and  $v_1$  belongs to three pentagons by Lemma 3.9 (impossible). Hence  $v_5$  is also incident to  $A$ . Using a similar discussion as the case  $v_1$  incident to  $h_1$  we can know this situation can

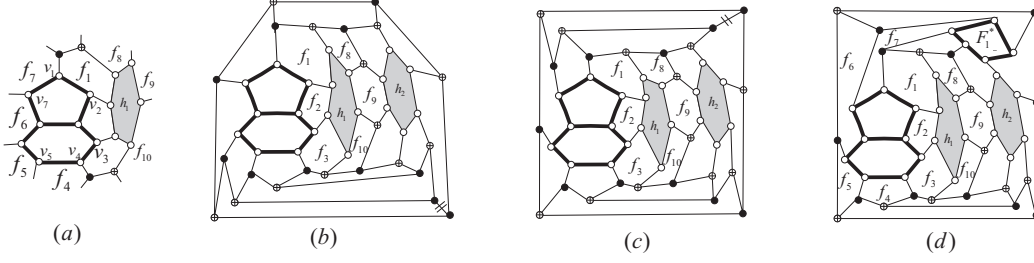


Figure 20. The case  $v_2, v_3$  incident to  $h_1$  and all of  $v_4, \dots, v_7$  incident to  $A$ .

not happen.  $\square$

**Claim 2:**  $F^*$  can not be isomorphic to the component of  $D08$  or  $D09$  or  $D11$  as illustrated in Fig. 5.

*Proof.* Also by contrary, suppose  $F^*$  is isomorphic to the component of  $D08$  or  $D09$  or  $D11$ . Let  $v_i$  and  $f_i$  ( $1 \leq i \leq 7$ ) be depicted in Fig. s 21(a), 22(a) and 22(d). Since  $F$  does not possess  $L, R$  as subgraphs,  $f_7$  is a hexagon when  $F^*$  is isomorphic to  $D08$  (see Fig. 21(a)) and  $f_2, f_3, f_5, f_6, f_7$  are hexagons when  $F^*$  is isomorphic to  $D09$  (see Fig. 22(a)) and  $f_1, f_3, f_5, f_7$  are hexagons when  $F^*$  is isomorphic to  $D11$  (see Fig. 22(d)). By Lemma 3.10 at least one of  $v_1, \dots, v_7$  is incident to  $h_1$  or  $h_2$ . Then similarly as Claim 1, firstly we suppose  $v_1$  is incident to  $h_1$  and we check that  $v_2, \dots, v_7$  are incident to  $H$  or not. If  $v_1$  is finished, then we let  $v_2$  be incident to  $h_1$  and test that  $v_3, \dots, v_7$  are incident to  $H$  or not. Execute the above procedure ceaselessly until all of  $v_1, \dots, v_7$  are incident to  $A$ . Using the above checking procedure as Claim 1 finally we can gain the fullerene graphs  $F_{46}^2, F_{46}^1, F_{48}^2, F_{48}^3$  as shown in Fig. 21(b),(c),(d),(e) when  $F^*$  is isomorphic to  $D08$  and  $F_{46}^4, F_{48}^4$  as shown in Fig. 22(b),(c) when  $F^*$  is isomorphic to  $D09$ . But these fullerene graphs are excluded in the assumption.  $\square$

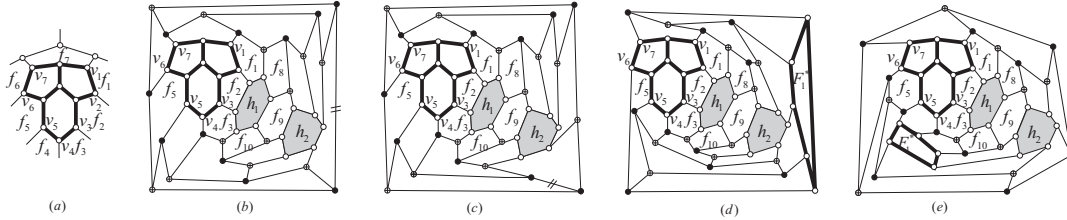


Figure 21. The case  $F^*$  being isomorphic to the component of  $D08$ .

**Claim 3:**  $\overline{F^*}$  can not contain a subgraph isomorphic to the component of  $ND03$  as shown in Fig. 16 (f)

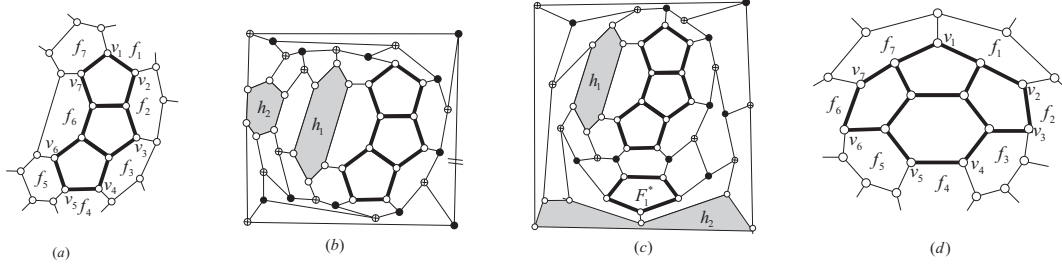


Figure 22. The case  $F^*$  being isomorphic to the component of  $D09$  or  $D11$ .

*Proof.* For convenience, we write  $G_1$  for the component of  $ND03$  shown in Fig. 16(f). By contrary suppose  $\overline{F^*}$  contains the subgraph  $G$ . Denote by  $f_i$  ( $1 \leq i \leq 5$ ) and  $P_i$  ( $1 \leq i \leq 6$ ) the hexagonal and pentagonal inner faces of  $G$  and  $f_j$  ( $6 \leq j \leq 12$ ) the six neighboring faces of  $G$  depicted in Fig. 23(a). If  $f_1$  equals  $h_1$ , then  $f_2$  contains at least one contributing edges by Lemma 3.3. That is, there are no other contributing edges except for the ones contained in  $f_2$  and  $E(F^*)$  as  $s(\mathcal{D}) \geq 2$ . In other words,  $P_6$  must contain a vertex in  $H$  by (\*). If  $f_3$  is the hexagon  $h_2$ , then  $f_4$  includes a contributing edge by Lemma 3.3, contradicting Ineq. (5) (see Fig. 23(a) the case  $f_4$  includes an  $E(A, H)$  edge). Similarly, none of  $f_4, f_5, f_6$  and  $f_7$  is the hexagon  $h_2$ , which means  $P_6$  contains no vertex in  $H$ , also a contradiction. Thus  $f_1$  does not equal  $h_1$ . So does  $f_2$  by symmetry. If  $f_3$  equals  $h_1$ , then similarly as the case  $f_1$  equals  $h_1$ ,  $f_4$  contains at least one contributing edge and  $P_1$  must include a vertex in  $H$ . However, no matter which face of  $f_1, f_5, f_8, f_9$  is the hexagon  $h_2$  we will obtain another contributing edge different from the one contained in  $f_4$ , again contradicting Ineq. (5). This contradiction means  $f_3$  can not be the hexagon  $h_1$ . So does  $f_5$  by symmetry. If  $f_4$  equals  $h_1$ , then exactly one of  $f_3$  and  $f_5$  intersects both  $h_1$  and  $h_2$ , otherwise, each of  $f_3$  and  $f_5$  contains a contributing edge by Lemma 3.3 and  $|E(A, H)| + |E(A, A)| + s(\mathcal{D}) \geq 4$  (impossible). Without loss of generality, suppose  $f_8$  equals  $h_2$  (see Fig. 23(b)), then  $f_9$  must be a hexagon, otherwise,  $P_1 \cup P_2 \cup P_3 \cup f_9$  forms a subgraph  $L$ , contradicting the assumption. Thus also  $f_9$  contains a contributing edge different from the one contained in  $f_3$ , which is impossible. So  $f_4$  does not equal  $h_1$ . Similarly,  $f_i$  ( $6 \leq i \leq 12$ ) cannot be the hexagon  $h_1$  or  $h_2$ . Since the only one possible non-trivial factor-critical component of  $F - H - A$  other than  $F^*$  is a pentagon by Prop. 3.2, whether  $P_i$  ( $1 \leq i \leq 6$ ) is the pentagonal factor-critical component or not we will obtain that  $|E(A, A)| + s(\mathcal{D}) \geq 4$  by applying (\*) to the six pentagons  $P_i$  ( $1 \leq i \leq 6$ ), also a contradiction.  $\square$



Figure 23. Illustration for Claim 3 in the proof of Lemma 3.13.

## 6 Proof of Lemma 3.14

Since every  $F^* \in \mathcal{D}$  with  $|\nabla(F^*)| = 5$  is a pentagon by Prop. 3.2 and  $1 \leq p(V(F^*)) \leq 3$  by Ineq. (8), we indicate the following three Claims hold to prove Lemma 3.14.

**Claim 1:** There is no a pentagonal component  $F^* \in \mathcal{D}^*$  such that  $p(V(F^*)) = 3$ .

*Proof.* Suppose to the contrary that there exists one such pentagonal factor-critical component  $F^*$ . Denote by  $v_1v_2v_3v_4v_5v_1$  the boundary of  $F^*$  along the clockwise direction and  $v'_i$  the neighbor of  $v_i$  ( $i = 1, 2, 3, 4, 5$ ) not in  $F^*$ . Let  $f_1, f_2, \dots, f_5$  be the five neighboring faces of  $F^*$  along the edges  $v_1v_2, v_2v_3, \dots, v_5v_1$ , respectively. Lemma 2.9 guarantees at most one of  $v_1, v_2, \dots, v_5$  is incident to  $h_j$  for some  $j \in \{1, 2\}$ . Note if there exists another  $F_1^* \in \mathcal{D}^*$ , then  $F_1^*$  is also a pentagon by Lemma 3.13 and the fact  $s(\mathcal{D}) \leq 3$ . We always do not distinguish  $h_1$  and  $h_2$ .

As  $p(V(F^*)) = 3$ , two of  $f_1, f_2, \dots, f_5$  are pentagons. Then the two pentagons are not adjacent by the assumption. By symmetry, we can suppose  $f_1$  and  $f_4$  are the two pentagons. Denote by  $f_6, f_7, \dots, f_{13}$  the faces and  $v_6, v_7, \dots, v_{13}$  the vertices of  $F$  as shown in Fig. 24(a). Then  $f_6, f_{12}$  are hexagons in order to prevent the forbidden subgraph  $L$  occurring in  $F$ .

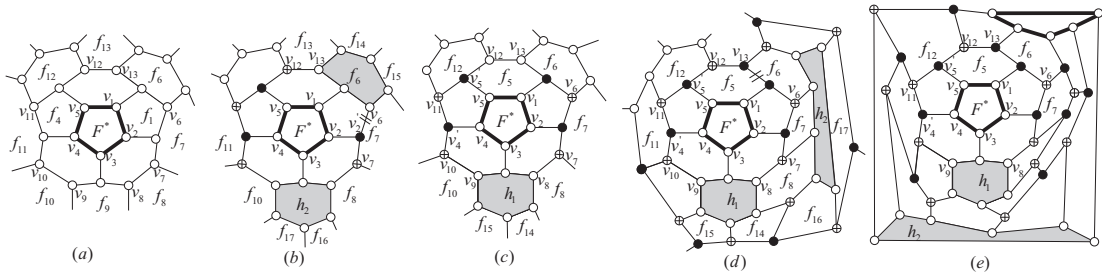


Figure 24. Illustration for Claim 1 in the proof of Lemma 3.14.

If  $v_1$  is incident to  $h_1$ , then  $f_1$  contains an  $E(A, H)$  edge  $v'_2v_6$  (see Fig. 24(b)). If  $v_3$  is incident to  $h_2$ , then  $v'_4, v'_5 \in A$  and  $v_7, v_{10}, v_{11}, v_{12} \in D_0$  by Ineqs. (5) and (6). Let  $f_{13}, f_{14}, f_{15}$  and  $f_{16}, f_{17}$  be the neighboring faces of  $h_1$  and  $h_2$  (respectively) as shown in Fig. 24(b). It's easy to see the neighboring faces of  $h_1$  and  $h_2$  are pairwise different. Thus at

least one of  $f_8, f_{10}, f_{16}, f_{17}$  contains a contributing edge by Lemma 3.3 in order to prevent the forbidden subgraph  $L$ . That is,  $p(V(h_2)) \leq 3$ . If this additional contributing edge belongs to  $E(h_1, h_2)$  or  $E(A, H)$ , then  $p(H) + p(E(A, A)) + p(D^*) \leq (4 + 3) + 3 = 10$  (impossible). If it belongs to  $E(A, A)$ , then  $p(H) + p(E(A, A)) + p(D^*) \leq (4 + 3) + 3 + 1 = 11$  (also impossible). If it belongs to  $E(F_1^*)$ , then  $F_1^*$  must be incident to both  $h_1$  and  $h_2$  as  $p(V(F_1^*)) \leq 3$ , which means  $p(V(h_2)) \leq 2, p(V(h_1)) \leq 3, p(V(F_1^*)) \leq 2$ . Hence  $p(H) + p(E(A, A)) + p(D^*) \leq (3 + 2) + (3 + 2) = 10$  (impossible). This contradiction implies  $v_3$  is incident to  $A$ . If  $v_4$  is incident to  $h_2$ , then  $f_4$  contains an  $E(A, H)$  edge  $v_{11}v'_5$  and  $f_2$  also contains a contributing edge belonging to  $E(A, A)$  or  $E(F_1^*)$  by Lemma 3.8. Thus  $|E(A, H)| + |E(A, A)| + s(\mathcal{D}) \geq 4$ , contradicting Ineq. (5). So  $v_4$  is also incident to  $A$ . If  $v_5$  is incident to  $h_2$ , then  $h_1$  and  $h_2$  are incident and each of  $f_2, f_3$  contains a contributing edge, which is impossible by Ineq. (6). Now all of  $v_2, \dots, v_5$  are incident to  $A$  and we once again obtain that  $|E(A, H)| + |E(A, A)| + s(\mathcal{D}) \geq 4$  (impossible). Thus  $v_1$  is incident to  $A$ . So does  $v_5$  by symmetry.

If  $v_2$  is incident to  $h_1$ , then  $f_1$  contains an  $E(A, H)$  edge  $v'_1v_6$  and we have a similar situation as the case  $v_1$  incident to  $h_1$ . The same analysis will deduce that for  $v_3$  or  $v_4$  incident to  $h_2$  or  $A$  we'll obtain  $p(H) + p(E(A, A)) + p(D^*) \leq (3 + 2) + (3 + 2) = 10$  (impossible) or  $|E(A, H)| + |E(A, A)| + s(\mathcal{D}) \geq 4$  (also impossible). Thus  $v_2$  is incident to  $A$ . Similarly for  $v_4$ .

If  $v_3$  is incident to  $h_1$ , then  $v_6, v_7, v_{10}, v_{11} \in D_0$  and  $f_5$  contains an  $E(A, A)$  or  $E(F_1^*)$  edge by Lemma 3.8. Denote by  $f_{14}, f_{15}$  the neighboring faces of  $h_1$  different from  $f_2, f_3, f_8, f_{10}$  (see Fig. 24(c)). Firstly suppose  $v'_1v_{13} \in E(A, A)$  (see Fig. 24(d)). By Ineqs. (5) and (6) there is at most one another  $E(h_1, h_2)$  or  $E(A, H)$  or  $E(A, A)$  edge or one  $F_1^* \in \mathcal{D}$  with  $|\nabla(F_1^*)| = 5$ . For none or one  $E(h_1, h_2)$  or one  $E(A, H)$  edge existence, we have  $p(H) + p(E(A, A)) + p(D^*) \leq (4 + 4) + 3 = 11$  (impossible). For one additional  $E(A, A)$  edge existence, say  $e \in E(A, A)$ , we obtain  $p(\{e\}) = 2$ , otherwise,  $p(H) + p(E(A, A)) + p(D^*) \leq (3 + 4) + 1 + 3 = 11$  (impossible). Moreover,  $f_6$  intersects  $h_2$  with the boundary  $HHD_0AAD_0$  by Observation 3.6 and  $e$  can not be contained in the neighboring faces of  $h_1$  and  $h_2$  by Lemma 3.5(1)(3). Thus one of  $f_8, f_{10}, f_{14}, f_{15}$  must intersect both  $h_1$  and  $h_2$  with the boundary  $HHD_0HHD_0$  and the remaining neighboring faces of  $h_1$  and  $h_2$  are pentagons with the boundaries  $HHD_0AD_0$  in order to avoid the occurring the forbidden subgraph  $L$ . From the above analysis we can construct the fullerene graph. It's easy to see the four neighboring faces ( $f_{14}, \dots, f_{17}$ ) of  $h_1$  and  $h_2$  form a subgraph  $L$  (see Fig. 24(d)) (impossible). For  $F_1^*$  existence,  $f_6$  must contain an  $E(F_1^*)$  edge and  $F_1^*$  must be incident to  $h_2$  but not  $h_1$ , otherwise, either it happens a subgraph  $R$  in  $F$  or  $p(H) + p(E(A, A)) + p(D^*) \leq 11$ , both

of which are impossible by the assumption and Ineq. (7). Hence the positions of  $h_2$  and  $F_1^*$  are known and we have the fullerene graph  $F_{46}^3$  as shown in Fig. 24(e), which is excluded in the assumption. Next we assume  $v_{12}, v_{13} \in V(F_2^*)$ . Then one of  $f_6, f_{12}$  intersects  $h_2$  and the remaining one contains an  $E(A, A)$  edge by Lemma 3.8. Without loss of generality, suppose  $f_6$  intersects  $h_2$ . Again four neighboring faces of  $h_1$  and  $h_2$  form a subgraph  $L$  (impossible). This contradiction means  $v_3$  is incident to  $A$ . Thus all of  $v_1, \dots, v_5$  are incident to  $A$  and by Lemma 3.8  $|E(A, A)| + s(\mathcal{D}) \geq 4$  (impossible).  $\square$

To make a summary, we can see  $p(V(F^*)) \leq 2$  for any  $F^* \in \mathcal{D}$  with  $|\nabla(F^*)| = 5$ .

**Claim 2:** There is no a pentagonal component  $F^* \in \mathcal{D}^*$  such that  $p(V(F^*)) = 2$ .

*Proof.* By contrary such a component  $F^*$  exists. Label the boundary of  $F^*$  and its neighboring faces as shown in Fig. 25(a). We also have if there exists another  $F_1^* \in \mathcal{D}^*$ , then  $F_1^*$  is also a pentagon. As  $p(V(F^*)) = 2$ , we can suppose  $f_1$  is pentagonal. Similarly as Claim 1, for  $v_2$  incident to  $h_1$  we have  $f_1$  contains an  $E(A, H)$  edge  $v'_1 v_6$  (see Fig. 25(b)) and  $v_3$  cannot be incident to  $h_2$ , otherwise,  $h_1$  and  $h_2$  are incident and  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (4 + 4) + 2 = 10$  (impossible). Moreover,  $v_4(v_5)$  also cannot be incident to  $h_2$ , otherwise,  $p(V(h_2)) \leq 3$  and  $f_5(f_3)$  contains an  $E(A, A)$  or  $E(F_1^*)$  edge by Lemma 3.8, but  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (4 + 3) + 2 + 2 = 11$  (impossible). So all of  $v'_3, v'_4, v'_5$  belong to  $A$  and  $|E(A, H)| + |E(A, A)| + s(\mathcal{D}) \geq 5$  (impossible). Hence  $v'_2 \in A$ . So does  $v'_1$  by symmetry and  $v_6 \in D_0$ . For  $v_3$  incident to  $h_1$ , then  $v_4$  cannot be incident to  $h_2$ , otherwise,  $h_1, h_2$  are incident and  $p(V(h_1)) \leq 3, p(V(h_2)) \leq 3$  and  $f_5$  contains an  $E(A, A)$  or  $E(F_1^*)$  edge. However, no matter which case occurs we'll have  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (3 + 3) + 2 + 2 = 10$  (impossible). If  $v_5$  is incident to  $h_2$ , then also  $p(V(h_1)) \leq 3, p(V(h_2)) \leq 3$ . Denote by  $f_6, f_7, f_8, f_9(f_{10}, f_{11}, f_{12}, f_{13})$  the four neighboring faces of  $h_1(h_2)$  different from  $f_2, f_3(f_4, f_5)$  (see Fig. 25(c)). Then the eight faces  $f_6, f_7, \dots, f_{13}$  are pairwise different by Lemma 2.10. Hence in order to prevent the occurring of the forbidden subgraph  $L$ , at least one of  $f_6, f_7, f_8, f_9$  contains a contributing edge by Lemma 3.3. Similarly for  $f_{10}, f_{11}, f_{12}, f_{13}$ . If these contributing edges belong to  $E(A, H)$  or  $E(A, A)$ , then at most two such edges exist in  $F$  and  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (3 + 3) + 2 + 2 = 10$  (Note an  $E(A, A)$  edge gives rise to one hexagon by Lemma 3.5(1)(3)) (impossible). So there must exist additional  $E(\mathcal{D}^*)$  edges (other than  $E(F^*)$ ), which means at least one of  $h_1, h_2$  (say  $h_1$ ) is incident to another pentagonal factor-critical component, say  $F_1^*$  (see Fig. 25(c)). Thus  $p(V(h_1)) \leq 2$  since the two common neighboring faces of  $h_1$  and  $F_1^*$  are hexagons by Lemma 3.8. On the other hand, if the contributing edge contained in  $f_{10}$  or  $f_{11}$  or  $f_{12}$  or  $f_{13}$  belongs to  $E(A, A)$  or  $E(A, H)$ , then we once again obtain that  $p(H) + p(E(A, A)) + p(\mathcal{D}^*) \leq (2 + 3) + 1 + (2 + 2) = 10$ ,



(impossible). If the contributing edge contained in  $f_j$  for some  $j \in \{10, 11, 12, 13\}$  belongs to  $F_2^* \in \mathcal{D}$  with  $|\nabla(F_2^*)| = 5$ , then similarly as the case above we have  $p(V(h_2)) \leq 2$ . If  $F_2^* = F_1^*$ , then  $p(H) + p(E(A, A)) + p(D^*) \leq (2 + 2) + 2 + (2 + 2) = 10$  (impossible). If  $F_2^* \neq F_1^*$ , then  $p(E(A, A)) = 0$  and  $p(H) + p(E(A, A)) + p(D^*) \leq (2 + 2) + (2 + 2 + 2) = 10$  (impossible). Hence  $v_5$  is also incident to  $A$ . Now both of  $f_4, f_5$  contain an  $E(A, A)$  or  $E(F_1^*)$  edge by Lemma 3.8(1). However, using a similar discussion as the cases above we can know no matter which case occurs we'll obtain  $p(H) + p(E(A, A)) + p(D^*) \leq 11$  (impossible). Thus  $v_3$  is incident to  $A$ . So does  $v_5$  by symmetry. Now both of  $f_2, f_5$  contain an  $E(A, A)$  or  $E(F_1^*)$  edge by Lemma 3.8 and for  $v_4$  incident to  $h_1$  we obtain  $p(H) + p(E(A, A)) + p(D^*) \leq 11$  (impossible) and for  $v_4$  incident to  $A$  we can gain  $|E(A, A)| + s(\mathcal{D}) \geq 4$  (also impossible).  $\square$

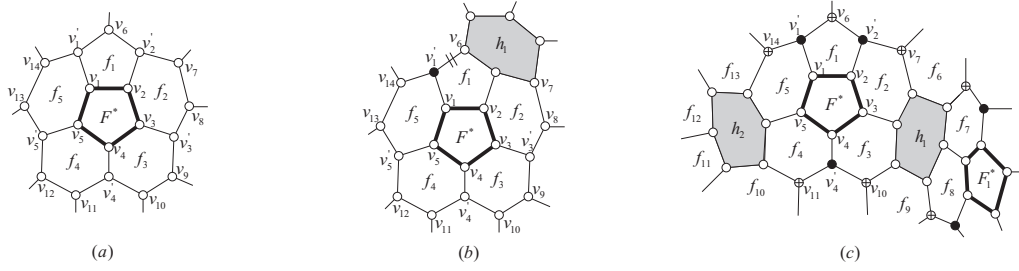


Figure 25. Illustration for Claim 2 in the proof of Lemma 3.14.

From the above analysis, we have  $p(V(F^*)) \leq 1$  for any  $F^* \in \mathcal{D}$  with  $|\nabla(F^*)| = 5$ .

**Claim 3:** There exists no a pentagonal component  $F^* \in \mathcal{D}^*$  such that  $p(V(F^*)) = 1$ .

*Proof.* Suppose there is one such component  $F^*$ . Also denote by  $v_1v_2v_3v_4v_5v_1$  the boundary of  $F^*$  and  $f_1, \dots, f_5$  the neighboring faces of  $F^*$  along the edges  $v_1v_2, v_2v_3, \dots, v_5v_1$ , respectively. Since  $p(V(F^*)) = 1$ ,  $f_i$  is hexagonal for all  $i \in \{1, 2, \dots, 5\}$ . Similarly as Claims 1, 2, for at most one of  $v_1, v_2, \dots, v_5$  incident to  $h_1$ , we obtain  $|E(A, A)| + s(\mathcal{D}) \geq 4$  (impossible). For two of  $v_1, v_2, \dots, v_5$  incident to  $h_1$  and  $h_2$  (respectively), we always have  $|E(A, A)| + s(\mathcal{D}) + |E(V(h_1), V(h_2))| \geq 4$  when the two vertices incident to  $h_1$  and  $h_2$  are adjacent on  $\partial(F^*)$  or  $p(H) + p(E(A, A)) + p(D^*) \leq 11$  when the two vertices incident to  $h_1$  and  $h_2$  are not adjacent on  $\partial(F^*)$ , which are impossible.  $\square$

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